# Asset Allocation with Multivariate Non-Gaussian Returns

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# OUTLINE

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- The Investment Horizon
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- Implementing Portfolio Theory with the VG Process
  - The VG Investor (one risky asset case)
  - The VG Asset Allocator (many risky assets)

# Motivation

- The objective is to deliver successful investment strategies in up to 50 stocks using a better understanding of the probability laws of returns based on Lévy processes.

- We have confirmed the superiority of Lévy processes to the Gaussian model in numerous studies of the statistical and risk neutral distributions.

- We expect that investment allocation stands to gain by employing this better understanding of returns.

# Some Economic Arguments

- It has long been recognized in economic theory that beyond mean and variance, the first two risk reward statistics there is a preference for skewness and an aversion to kurtosis.

- We can also construct many portfolios with the same mean and return variance structure that differ in the structure of the higher moments and the investor is not expected to be indifferent between these portfolios.

- For these reasons we wish to capture at least the first four moments in the design of investent strategies.

# Information Theoretic Arguments

- With full information we have the resolution of uncertainty and the outcome is a delta function.

- With some uncertainty one may expect some entropy and a density with a long neck and some tails or a typical leptokurtic density.

- For zero mean, unit variance maximal uncertainty occurs at the Gaussian random variable and hence in some sense, the Gaussian model is investing on the assumption of zero information.

# The Investment Horizon

- Professional investors can reevaluate positions at a much greater frequency and are often engaged in what are short horizons.

- This is good for the Lévy process perspective as the homogeneity assumption is relatively easy to maintain over the short horizon.

- In the longer term we do have movements in the return characteristics and have to cope with stochasticity in volatility, skewness and kurtosis.

# The Difficulties

- The first difficulty in implementing such a plan is the description and estimation of multidimensional probability laws.
  - Most of our univariate estimation exploits one dimensional fast Fourier inversion of characteristic functions.
  - The multidimensional counterpart is still well beyond our computational abilities.

- The second difficulty is the issue of data requirements
  - Even in the Gaussian case one runs out of data if one attempts to estimate very large covariance matrices.
  - Even with the 50 stocks the total number of parameters to be estimated is quite large if we allow for a full covariance structure.

- The third difficulty is the solution for the optimal portfolio in such a multidimensional context.
  - One may attempt to maximize the simulated expected utility, if one has learned to simulate from the multidimensional law.
  - Such an approach would be very expensive to back test and without the back test there is little hope to be funded for a front test.

# A Proposed Solution

- We reduce both the estimation and investment problems to one dimensional problems.

- These are solved efficiently and are easily implemented using the methods already developed in the derivatives literature.

- We performed a back test over 125 non-overlapping monthly investments.

# Strategy

- We postulate returns as a linear mixture of independent Lévy processes called factors.

- We employ Independent Components Analysis (ICA) to identify the mixing matrix and to extract data on the factors.

- The ICA analysis produces the large covariance structure.

- We use univariate methods to estimate the Lévy processes on the factors one at a time.

- We reduce the problem of investment in assets to a sequence of problems of optimal investment in the factors taken individually.

- These are solved in closed form and the asset investment is then inferred from the already identified mixing matrix.

# The VG Process

- The process may be defined from standard Brownian motion $(W(t), t \geq 0)$ allowing a drift $\theta$ and volatility $\sigma$ to form

$$B(t; \theta, \sigma) = \theta t + \sigma W(t)$$

- We now introduce an independent gamma process

$$(G(t; \nu), t \geq 0)$$

  with mean rate unity and variance rate $\nu$ with density for the increment $g = G(t+h) - G(t)$ given by the gamma density

$$f(g) = \frac{g^{\frac{h}{\nu}-1} e^{-\frac{g}{\nu}}}{\nu^{\frac{h}{\nu}} \Gamma\left(\frac{h}{\nu}\right)}.$$

  as a model for a random time change.

# The VG Process

- The three parameter $VG$ process $(X(t), t \geq 0)$ is then obtained as a time changed Brownian motion with drift and we have

$$\begin{aligned} X(t) &= B(G(t;\nu);\theta,\sigma) \\ &= \theta G(t;\nu) + \sigma W(G(t;\nu)) \end{aligned}$$

- This process is quite analytic and tractable in many of its dimensions and the three parameters allow one to capture movements in skewness and kurtosis as well as volatility.

# The VG Characteristic Function

- The characteristic function is particularly analytic and is easily evaluated by first conditioning on the level of the gamma time change and then using the Laplace transform of the gamma density to get

$$\begin{aligned} E\left[e^{iuX(t)}\right] &= \phi_{X(t)}(u) \\ &= \left(\frac{1}{1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2}\right)^{\frac{t}{\nu}} \end{aligned}$$

- Inversion using the Fast Fourier transform quickly gives access to the density that may be employed in maximum likelihood estimation.

# Process Properties

- We learn from the Lévy Khintchine decomposition of the characteristic function that the process is one of finite variation, in fact it can be written as the difference of two gamma processes each describing separately the process for the market upticks and down ticks.

- Explicitly we have

$$\phi_{X(t)}(u) = \exp\left(\int_{-\infty}^{\infty}\left(e^{iux} - 1\right) k_{VG}(x)dx\right)$$

where the Lévy measure $k_{VG}(x)$ takes the form

$$\begin{aligned} k_{VG}(x) &= C\left[\frac{e^{-Ax-B|x|}}{|x|}\right] \\ A &= \frac{G-M}{2} \\ B &= \frac{G+M}{2} \end{aligned}$$

# The Early Successes

- The first successes with these processes showed that they were very capable of explaining the unconditional return densities on asset prices.

- The basic model employed for the stock price $S(t)$ as driven by $VG$ or $CGMY$ was

$$S(t) = S(0)\exp\left((\mu + \omega)t + X(t)\right)$$

  where $\omega$ is the convexity correction defined by

$$Ee^{X(t)} = e^{-\omega t}.$$

- I present here a graph for $VG$ on the $SPX$ and a Chi-Square Table for the major indices.

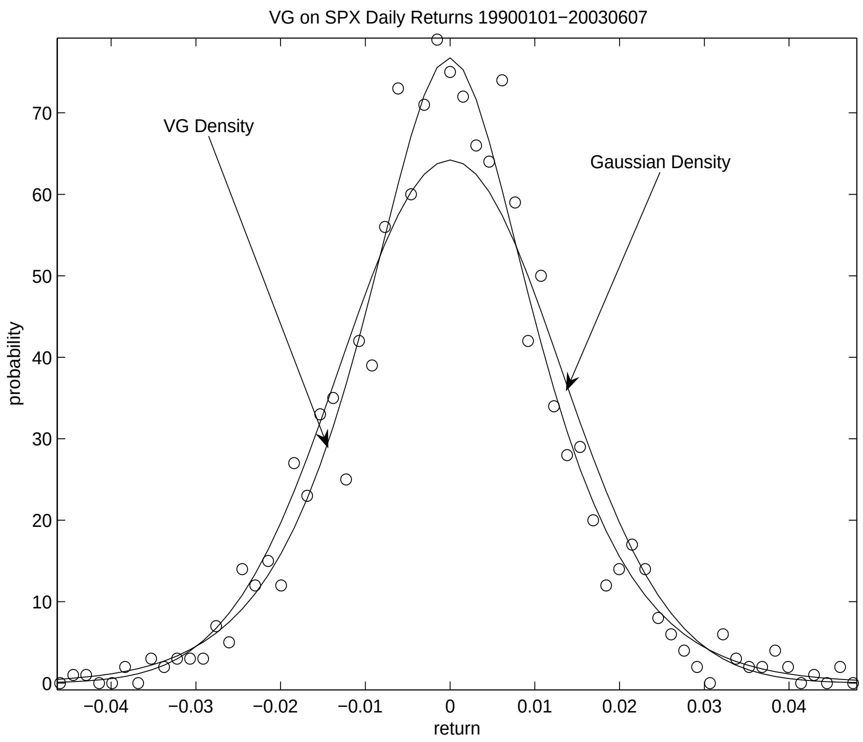



Figure 1: Fit of VG to SPX 19900101 t0 20040607

TABLE 1

Statistical Estimation

| | SPX | DAX | FTSE | IBEX | NIKKEI |
|---|---|---|---|---|---|
| volatility | .1679 | .2569 | .1718 | .2222 | .2445 |
| $\sigma$ | .1662 | .2545 | .1691 | .2202 | .2424 |
| $\nu$ | .0034 | .0031 | .0024 | .0022 | .0024 |
| $\theta$ | -.0447 | -.4548 | -.0765 | -.3502 | .0610 |
| $C$ | 13.02 | 23.04 | .2927 | 2.79 | 5.11 |
| $G$ | 94.64 | 65.24 | 51.99 | 63.08 | 68.57 |
| $M$ | 100.2 | 78.10 | 56.37 | 75.60 | 66.16 |
| $Y$ | .5348 | .4925 | 1.21 | .8963 | .7982 |
| chisq Gauss | 463.5 | 213.9 | 211.9 | 132.7 | 168.1 |
| chisq VG | 47.9 | 49.4 | 65.3 | 35.7 | 46.2 |
| chisq CGMY | 42.0 | 49.8 | 48.8 | 32.2 | 47.3 |
| pval Gauss | 0 | 0 | 0 | 0 | 0 |
| pval VG | 8.82% | 41.7% | 0.2% | 81% | 54.7% |
| pval CGMY | 22.6% | 40.3% | 7.5% | 90.6% | 50.2% |

# Risk Neutral Distributions

- The success with calibrating the prices of options has been equally good. For multiple maturities we recognized as stated earlier that a Lévy process was inadequate but extensions incorporating stochastic volatility are described in Schoutens (2003). I present here the $CGMYSA$ christmas tree.

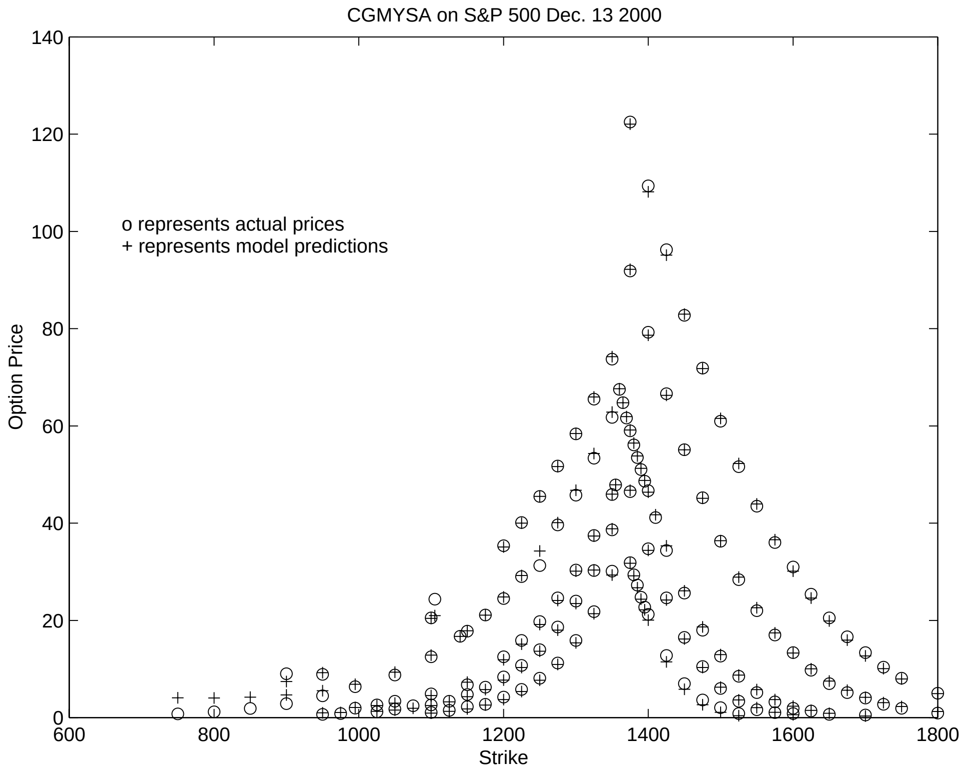

CGMYSA on S&P 500 Dec. 13 2000
o represents actual prices
+ represents model predictions
Option Price
Strike
0
20
40
60
80
100
120
140
600
800
1000
1200
1400
1600
1800

Figure 2:

# Single Risky VG Asset

- Consider a short period risky return $R$ in which we may invest $y$ dollars on a financed basis to earn the final wealth of

$$W = y(R - r)$$

- If we suppose that $R$ is distributed as

$$R = \mu + Z$$

where $Z$ Gaussian with variance $\sigma^2$ and mean 0 then the certainty equivalent for exponential utility and risk aversion $\rho$ is

$$CE = (\mu - r)y - \frac{y^2}{2}\rho\sigma^2$$

- The solution for the optimal short horizon investment $y^*$ is

$$y^*_{Gauss} = \frac{\mu - r}{\rho\sigma^2}.$$

- If we now take $Z$ to be a zero mean $VG$ random variable we get the formulation

$$Z = \theta(g-1) + \sigma W(g)$$

where $g$ has the gamma density with variance $\nu$ and unit mean.

- The certainty equivalent is now

$$CE = (\mu - r - \theta)\, y + \frac{1}{\nu\rho} \ln\left(1 + \nu\theta\rho y - \frac{\sigma^2\nu}{2}\rho^2 y^2\right)$$

# Lévy Certainty Equivalent

- More generally for the utility function

$$U(W) = 1 - \mathsf{exp}(-\rho W)$$

- With investment return

$$y(\mu - r) + yZ$$

- and characteristic exponent

$$E\left[e^{iuZ}\right] = \mathsf{exp}(-\psi_Z(u))$$

- We have

$$CE = y(\mu - r) + \frac{\psi_Z(i\rho y)}{\rho}$$

# The VG Investor

- The optimal investment by exponential utility with risk aversion $\rho$ in the $VG$ risky asset is

$$\rho y^*_{VG} = \left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right) + sign(\mu - r) \times$$

$$\left[\left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\,\nu}\right)^2 + \frac{2(\mu - r)}{(\mu - r - \theta)\,\nu\sigma^2}\right]^{\frac{1}{2}}$$

- The investment is long when $\mu > r$ and short otherwise and scales, like the Gaussian investment, inversely with the level of risk aversion.

- More generally, exponential utility and all completely monotone utilities display skewness preference along with volatility and kurtosis aversion with the investment being positively related to $\theta$ and negatively related to $\sigma, \nu$.

# The VG Asset Allocator

- The more interesting question is how to spreads investment dollars across a vector of risky assets with a view to accessing some diversification benefits.

- For this purpose the vector of risky assets is modeled as having a unit period return of $R$ with mean $\mu$ and a zero mean random component $Z$ with

$$R = \mu + Z$$

- Now we incorporate dependence by assuming that there exist independent random variables $X$ in principle equal in number to the dimension of $R$ such that

$$Z = AX.$$

# A Gaussian Review

- If we take $X$ to be independent Gaussian random variables then we arrive at the Markowitz investment solution with zero interest rates of

$$y^*_{Gauss} = \frac{1}{\rho}\Sigma^{-1}(\mu - r)$$

for risk aversion $\rho$ and return covariance matrix $\Sigma$.

- This is now quite easily implemented for 50 stocks.

- There are problems of data reqirements and matrix inversion if the number of stocks is expanded to say a 1000.

# Gaussian Solution as a Sequence of Univariate Problems

- A principal components analysis decomposition of the covariance matrix $\Sigma$ shows that on writing

$$\Sigma = UDU'$$

  for $U$ orthonormal we have factor excess returns

$$F = U'(R - r)$$

- With factor variances given by

$$E[FF'] = U'\Sigma U = D$$

- The factor investments are as for the univariate problem factor excess return scaled by risk aversion and variance

$$\widetilde{y}_i^* = \frac{[U'(\mu - r)]_i}{\rho D_{ii}}$$

- The asset investment is the one implied by the optimal univariate factor investment

$$
\begin{aligned}
y^*_{Gauss} &= U\widetilde{y}^* \\
&= \frac{1}{\rho} U D^{-1} U' (\mu - r) \\
&= \frac{1}{\rho} \left[ U D U' \right]^{-1} (\mu - r) \\
&= \frac{1}{\rho} \boldsymbol{\Sigma}^{-1} (\mu - r)
\end{aligned}
$$

# Using VG Factors

- For the $VG$ case we suppose that the independent original random variables that were mixed to form the asset returns all have their own $VG$ law as per

$$X_i = \theta_i(g_i - 1) + \sigma_i W_i(g_i)$$

  where the $g_i's$ are independent gamma variates and the $W_i$ are independent Brownian motions.

- The question arises as to how to recover these hidden $VG$ processes from data on the vector of asset returns.

- Given the $VG$ processes can we determine the optimal dollar allocation across assets.

# Lévy Asset Allocation

- For exponential utility and an independent Lévy factor structure the allocation problem splits into univariate problems with optimal factor investment $\widetilde{y}^*$.

- The certainty equivalent is now

$$CE = y'(\mu - r) + \frac{1}{\rho}\sum_j \psi_j(i\rho(y'A)_j)$$

- We write this in terms of factor investments

$$\widetilde{y}' = y'A$$

- as

$$CE = \widetilde{y}'\left(A^{-1}(\mu - r)\right) + \frac{1}{\rho}\sum_j \psi_j(i\rho\widetilde{y}_j)$$

- Defining factor excess returns by

$$\zeta = A^{-1}(\mu - r)$$

- We have the sequence of univariate problems

$$CE = \sum_j \zeta_j \widetilde{y}_j + \frac{1}{\rho}\psi_j(i\rho\widetilde{y}_j)$$

- The certainty equivalent in terms of $\widetilde{y}_i$ is additive with

$$
\begin{aligned}
CE &= \sum_{i=1}^{n} \left( \zeta_i \widetilde{y}_i + \frac{1}{\rho \nu_i} \ln \left( 1 + \theta_i \nu_i \widetilde{y}_i - \frac{\sigma_i^2 \nu_i}{2} \rho^2 \widetilde{y}_i^2 \right) \right) \\
\zeta &= \frac{1}{\rho} [A^{-1}(\mu - r) - \theta] \\
y_{VG}^* &= \frac{1}{\rho} A^{-1} \widetilde{y}
\end{aligned}
$$

- We solve for the univariate $VG$ factor investment as in the $VG$ investor and then obtain the allocation across assets using the inverse factor structure matrix.

# Identifying VG Factors using ICA

- We employ Independent Components Analysis, a statistical methodology aimed at identifying independent components from data obtained as an unknown linear mixture of component readings.

- The methodology is viewed as a generalization of $PCA$ or principal components as it begins by recognizing that principal components are determined only up to a rotation and all rotations yield the same covariance structure.

- It is also recognized that the Gaussian density is one that maximizes entropy among all densities on the line with zero mean and unit variance.

- ICA operates on the assumption that the original signals or factor sources have information, are therefore non-Gaussian and display excess kurtosis.

- It is then suggested that mixing leptokurtotic signals linearly reduces kurtosis and the strategy for identifying the original factors is that of successively maximizing the presumed excess kurtosis.

- A more robust criterion suggested is that of maximizing the expected logarithm of the hyperbolic cosine of the proposed factor data.

# The ICA Procedure

- A good text book presentation is given in Hyvärinen, Karhunen and Oja (2001), Wiley.

- Center the data to zero mean to form

$$\widetilde{R} = R - \mu$$

- Perform a PCA step and diagonalize the covariance matrix and write

$$\mathbf{\Sigma} = VV'$$

- Define the zero mean unit variance orthogonal random variables

$$H = V'\widetilde{R}.$$

- Recognize that for all orthonormal matrices $U$ we must have that $UH$ is still another zero mean, unit variance, orthogonal set of random variables.

- Find the first column of the matrix $U$ as on the unit circle with maximal expected log cosh.

- Find the next column as on the unit circle and orthogonal to the first column, and still maximizing expected log cosh.

- Continue till all $n$ columns of $U$ have been determined.

- The ICADATA is

$$\begin{aligned} X &= UV'\widetilde{R} \\ A &= (UV')^{-1} \end{aligned}$$

# Estimating VG Parameters

- We now estimate by univariate maximum likelihood estimation the $VG$ parameters for each of the factors.

- We may reduce the number of long tailed factors and treat as idiosyncratic all the components with a zero excess kurtosis or a factor density that is Gaussian.

- We view leptokurtoticity as tantamount to the presence of information and accept as noise the Gaussian components.

# Comparison of VG and Gaussian Investments

- Having estimated the $VG$ parameters we build the $VG$ dollar investments across the asset space and compare the results with those obtained on the Gaussian strategy.

- We presents results for data on 35 stocks that are prominent in the S&P 500 index.

- We present graphs of the $VG$ density fits for the first three $ica$ factors.

- We present a graph of the Chi-Square statistic on all the $ica$ factors to judge the number of factors needed before we can get down to Gaussian components.

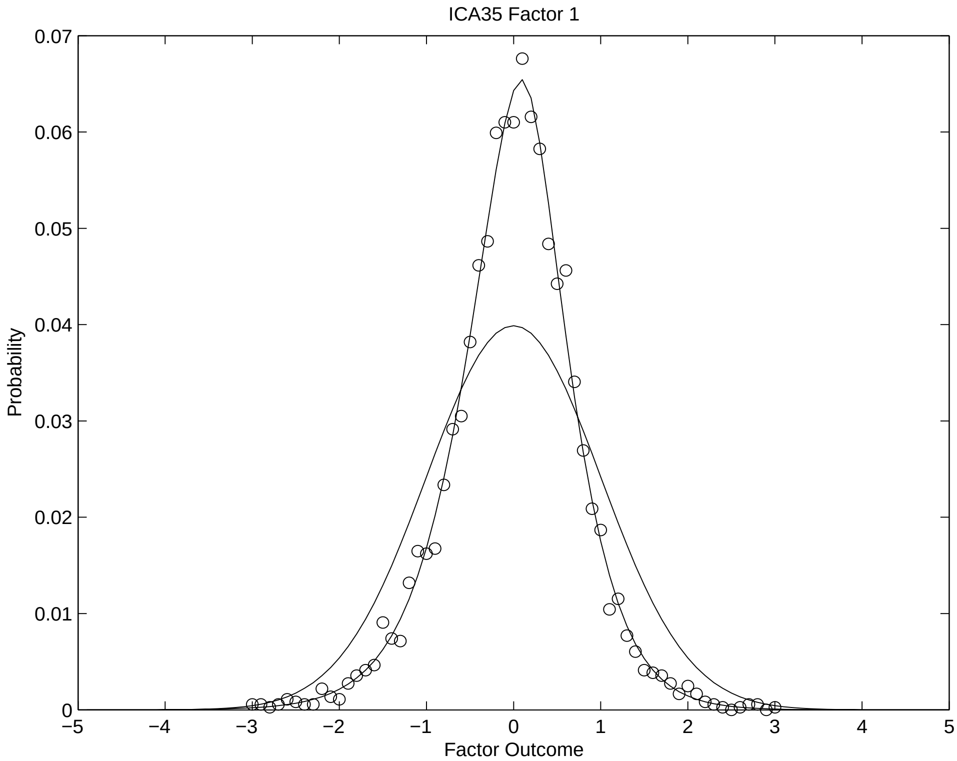
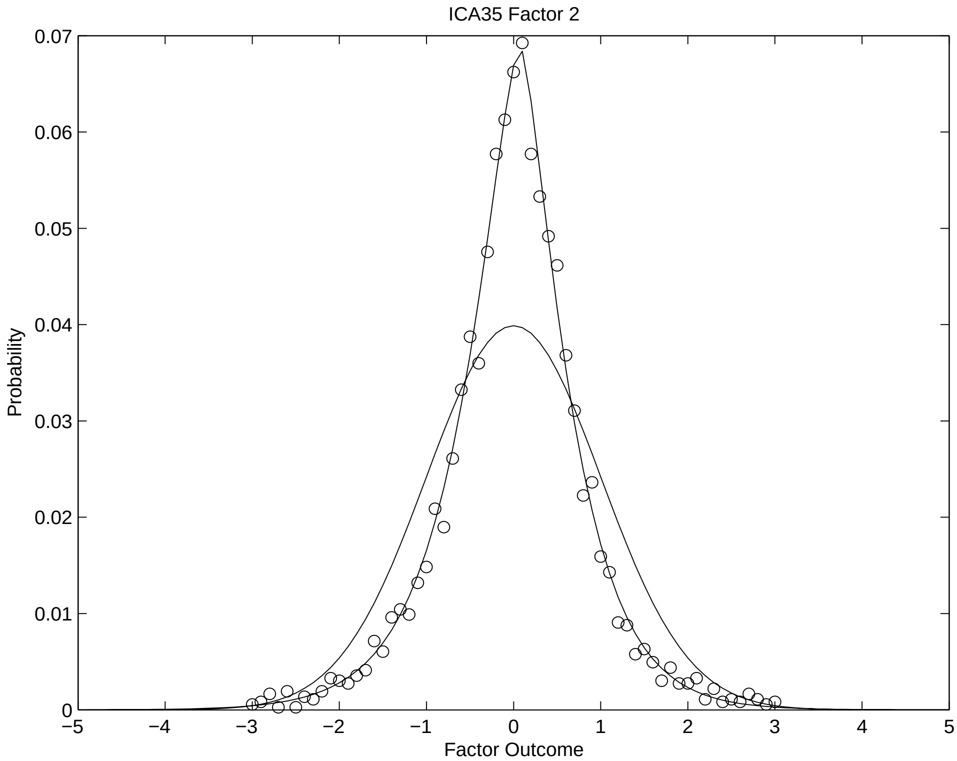

ICA35 Factor 1
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Probability
−5
−4
−3
−2
−1
0
1
2
3
4
5
Factor Outcome

Figure 3:



ICA35 Factor 2
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Probability
−5
−4
−3
−2
−1
0
1
2
3
4
5
Factor Outcome


Figure 4:

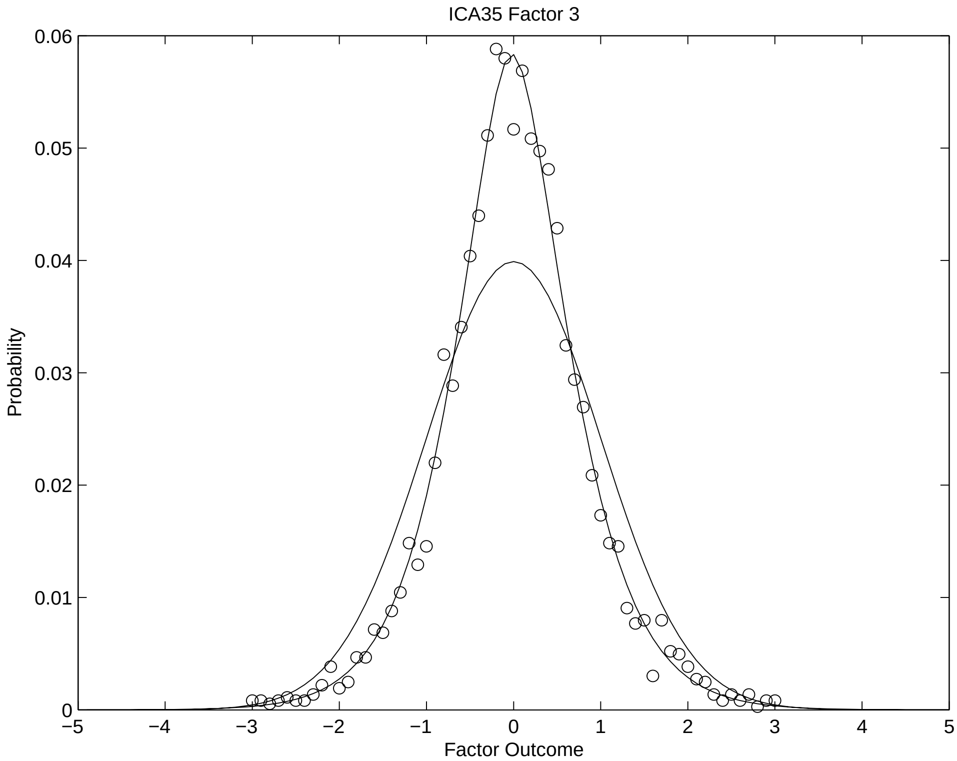

ICA35 Factor 3
Probability
Factor Outcome
0.06
0.05
0.04
0.03
0.02
0.01
0
−5
−4
−3
−2
−1
0
1
2
3
4
5

Figure 5:

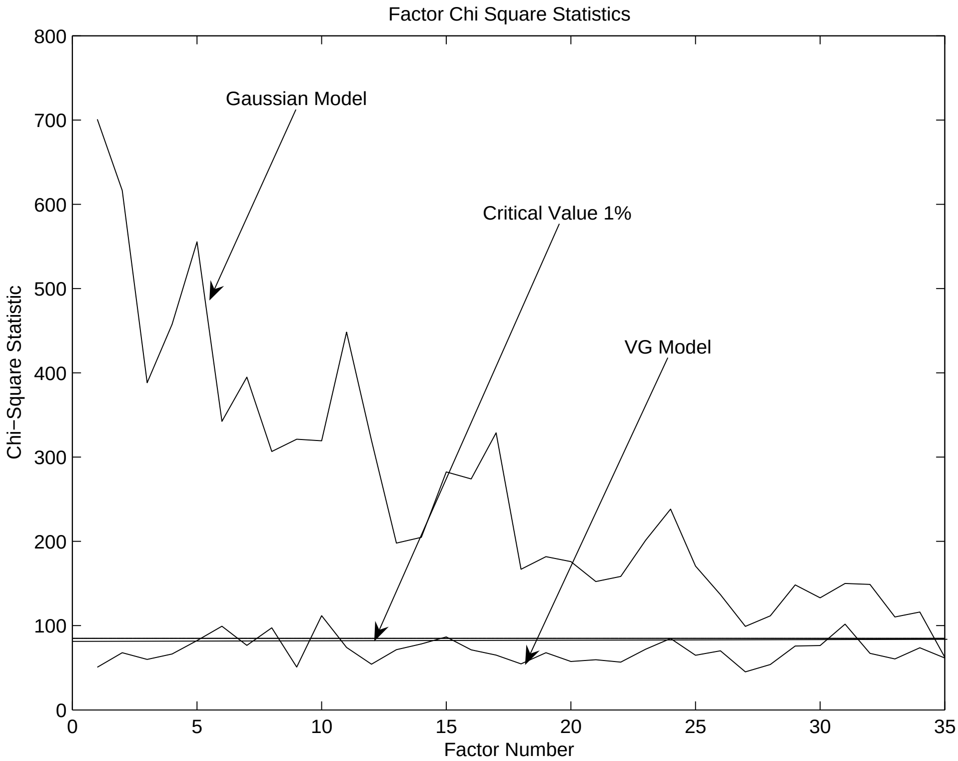

Factor Chi Square Statistics
Gaussian Model
Critical Value 1%
VG Model
Chi−Square Statistic
Factor Number
0
100
200
300
400
500
600
700
800
0
5
10
15
20
25
30
35

Figure 6:

# Back Testing the VG Asset Allocator

- With a view to getting ready for this meeting we studied the VG Asset Allocator's investments over a period of 125 non-overlapping intervals of 21 days.

- At the start of each 21 day period the asset allocator gets the interest rate prevailing for this period and determines the amount of dollars to be invested in each of five stocks.

- The allocator is fully financed and can invest as much as it wants, long or short.

- The allocator uses return data from the previous 1000 days to estimate the covariance matrix and mean returns for a Gaussian investment.

- For a VG allocation, we perform the $ICA$ analysis to get the $ICA$ data, then employ $MLE$ to get the parameters for the $VG$ model estimated on each of the factors using univariate methods. Finally, we use our analytical expressions for the optimal $VG$ investment in the five stocks MMM, BA, IBM, JNJ, and MCD.

- The period studied is December 1993 to April 2004.

# Investment Levels

- The mean investment levels in dollars were

Mean Investment Levels

| | MMM | BA | IBM | JNJ | MCD |
|---|---|---|---|---|---|
| Gauss | 0.5428 | 0.1142 | 0.3402 | 0.6357 | 0.5209 |
| VG | 255.16 | 91.34 | 150.17 | 281.68 | 193.68 |

- The VG investment is also more variable with greater frequency of short positions

Investment Standard Deviation

| | MMM | BA | IBM | JNJ | MCD |
|---|---|---|---|---|---|
| Gauss | 0.3387 | 0.4174 | 0.3260 | 0.3391 | 0.4891 |
| VG | 1301 | 1265 | 1224 | 1776 | 1440 |

# Performance Measures

- We report Sharpe Ratios, Gain Loss Ratios, and Certainty Equivalents for risk aversion .005 on the aggregate investment performance

| | VG | Gauss |
|---|---|---|
| Sharpe Ratio | 0.8757 | 0.7543 |
| Gain Loss Ratio | 2.3909 | 1.4536 |
| CE | 3.0539 | 0.0230 |

- We close with graphs of the cumulated cash flows, and the CDF of cash flows.

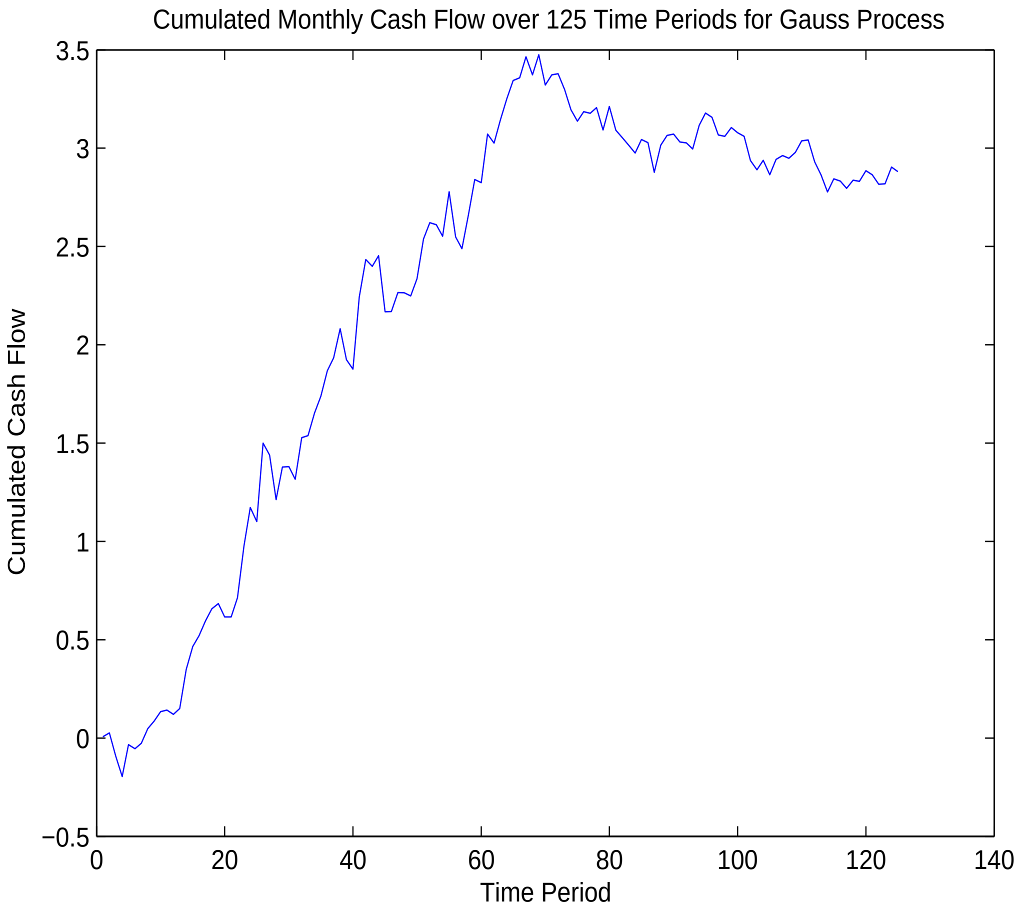
# Conclusion

- The variance gamma process in particular and Lévy processes more generally provide an interesting and desirable alternative to Brownian motion as a model for the local motion.

- They model in addition to volatility, important components of the distribution of returns, both statistically and risk neutrally.

- Many tractable improvements in model performance can be, and have been accessed using these models.

- The role played by independent components analysis is possibly interesting.



Cumulated Monthly Cash Flow over 125 Time Periods for Gauss Process
Cumulated Cash Flow
Time Period
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
0
20
40
60
80
100
120
140


Figure 7:

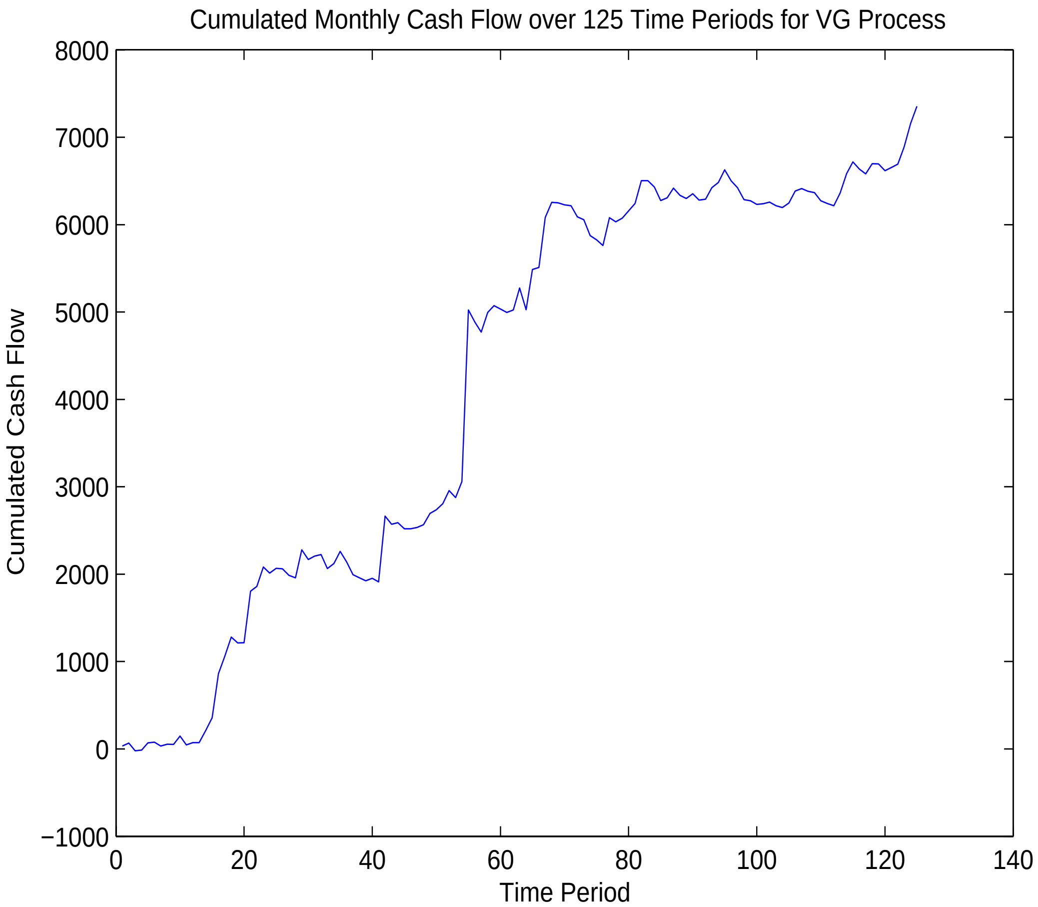

Cumulated Monthly Cash Flow over 125 Time Periods for VG Process
8000
7000
6000
5000
4000
3000
2000
1000
0
−1000
0
20
40
60
80
100
120
140
Time Period
Cumulated Cash Flow

Figure 8:

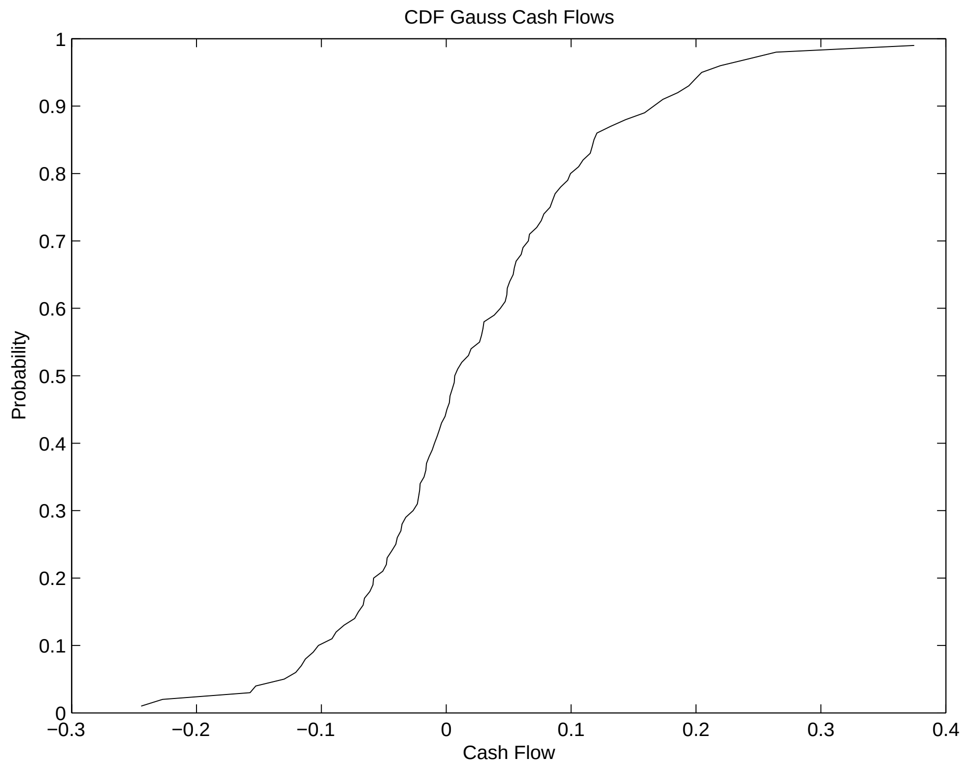

CDF Gauss Cash Flows
Probability
Cash Flow
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
0.4

Figure 9:

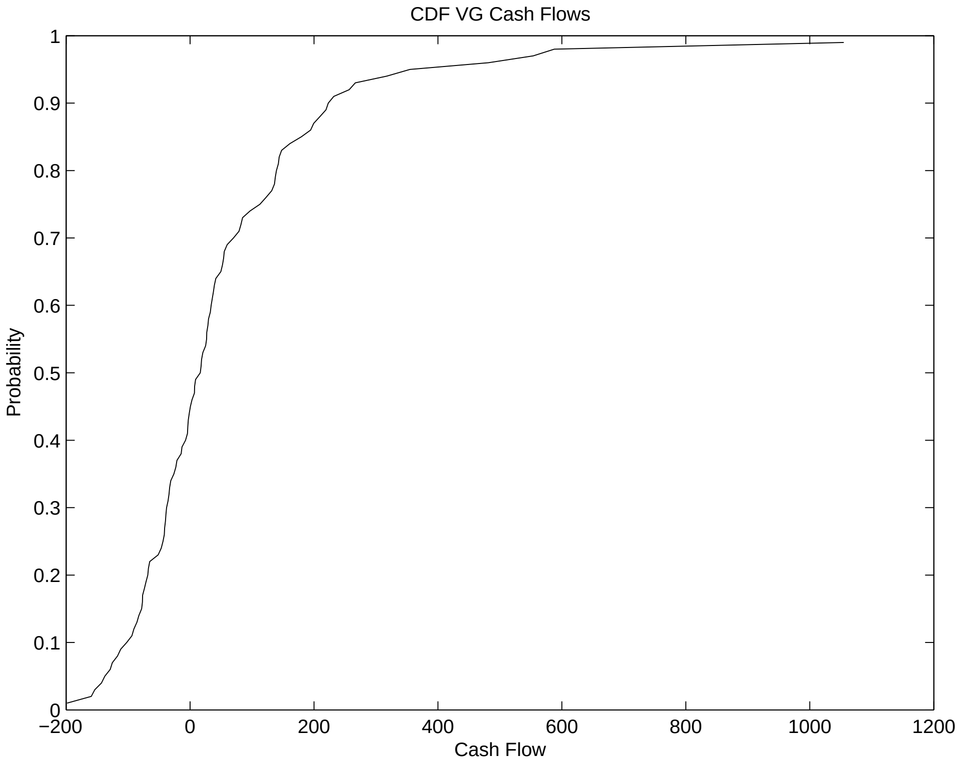

CDF VG Cash Flows
Probability
Cash Flow
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−200
0
200
400
600
800
1000
1200

Figure 10: