Cross-market valuation

This article takes the guesswork out of what credit margin to use when valuing credit-risky derivatives, and also sheds light on how relative value trading and capital structure arbitrage may be analysed quantitatively. Doug Moss builds a model up from the basics and shows the results of its application to France Telecom, a company with a very complex capital structure, and also applies it to a traditionally difficult area of valuation, that of convertible bonds

Given the task of valuing credit-risky liabilities, the dominant frameworks are the structural corporate debt models following Merton (1974), and the reduced-form models following Jarrow & Turnbull (1995) and Duffie & Singleton (1999). Structural models use capital structure and some estimate of the volatility of the company's assets to dictate the likeliness and timing of debt payment defaults. Classic versions of the reduced-form model do not use capital structure, and defaults depend instead upon a random process, the parameters of which are in practice largely determined by regressions of historical and current data for the market and company, possibly including both equity and bond price data.

Reduced-form models can therefore be expected to characterise default probabilities and credit margin term structures quite well, but say little about the relative value of a limited liability company's different liabilities (for example, debt, equity, convertibles, hybrids), which is the primary focus of this article. Clearly, capital structure is important for this task and therefore our framework extends Merton's structural model.

Historical models review

Merton's risky debt model can be summarised as follows:

☐ Define the firm's assets, equity and debt to be A, E and D respectively. Assume:

$$A = E + D \qquad (1)$$

☐ Define D_F as the debt face promised at maturity, T.

☐ The value of debt, D_T, repaid at T can be no greater than the assets:

$$\begin{aligned} &D_T = \min\{D_F, A_T\}. \text{ This can be rewritten as:} \\ &D_T = D_F - \max\{D_F - A_T, 0\}, \text{ or:} \\ &D_T = \text{risk-free bond}(D_F) \text{ minus a put on assets}(A_T) \text{ with strike}(D_F) \end{aligned} \qquad (2)$$

The put represents a credit risk premium paid to the lender to compensate for the chance of bankruptcy and the resultant repayment of only part of the debt.

☐ The value of equity, E_T, at T is asset value less debt value:

$$\begin{aligned} &E_T = A_T - D_T = A_T - \min\{D_F, A_T\}. \text{ This can be rewritten as:} \\ &E_T = \max\{A_T - D_F, 0\}, \text{ or:} \\ &E_T = \text{a call on assets}(A_T) \text{ with strike}(D_F) \end{aligned} \qquad (3)$$

☐ Assume the firm's assets have Black-Scholes-type volatility for valuation purposes:

$$\ln A_\tau \sim N\left\{\ln A + \left(r - \sigma^2/2\right)\tau, \sigma\tau^{1/2}\right\} \qquad (4)$$

where τ is elapsed time, r is the 'risk-free' rate, σ is the volatility of A and $N\{m, sd\}$ means normally distributed with mean m and standard deviation sd. This corresponds to a risk-neutral lognormal process with drift $(r - \sigma^2/2)\tau$.

☐ Therefore, the present value of equity and the credit risk premium is given by the Black-Scholes equations for a vanilla European-style call and put respectively.

However, the 'classic' Merton model is an oversimplification, and typically (see Gemmill, 2002) overvalues debt relative to the market, particularly with lowly leveraged and/or short-dated debt. Geske (1977) extended the Merton model to multi-maturity and subordination, and Black & Cox (1976) added debt covenants via a fixed knock-out barrier. Tax effects and bankruptcy costs were added by Leland (1994), and volatile interest rates by Longstaff & Schwartz (1995). Crosbie (1997) infers asset volatility from the historical share price, and also describes a mapping of the model-based risk-neutral first passage probability for crossing the barrier into an 'expected default probability' based on corporate default history. Finklestein *et al* (2002) use a 'shifted-lognormal' process with a 'fuzzy' barrier that partly addresses the issue of underestimating short-dated credit spreads. They also make use of leverage-induced volatility skews. Sousa (2002) uses a binomial framework that extends the treatment of guarantees, subordination and debt/equity hybrids, and calculates the value and the total risk of portfolios of debts of different issuers by correlating them through their capital asset pricing model betas. Hanke (2004) builds on Ericsson & Reneby's (2001) probabilistic extensions of Merton's model to give analytic solutions for a number of corporate-linked contingent claims including European-style equity option prices.

Before describing our extensions, we provide graphs to illustrate the above. Equation (2) implies that the credit margin required of corporate borrowers equates to the value of a put option, so we could expect to see a correlation between credit margins and equity volatility, particularly for a highly leveraged company such as France Telecom. Of course, the complete story of figure 1 is much more complex than simply a volatility-driven put premium, but the graph is nonetheless suggestive.

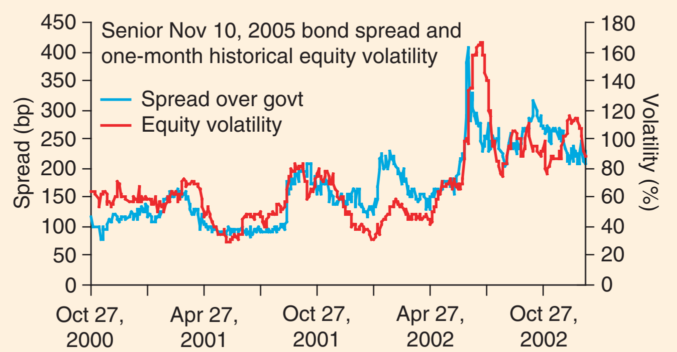
Equation (3) implies that equity is a call option. There is a well-known relationship (see Cox & Rubinstein, 1984) between the instantaneous ('local') volatility of a call option, σ_E, and its delta, Δ_E, and the volatility of its underlying, σ_A:

$$\sigma_E = A/E\,\Delta_E\sigma_A \qquad (5)$$

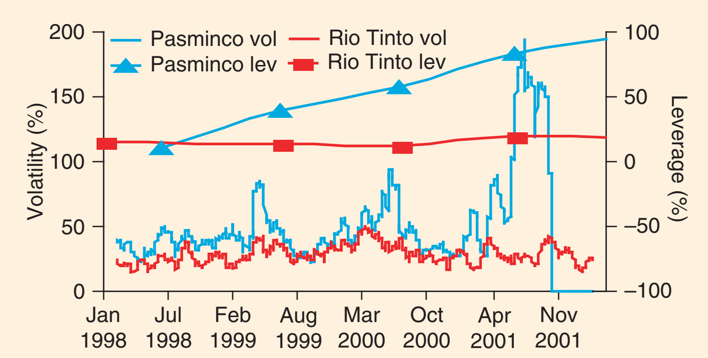
from which it follows that equity volatility increases with leverage.

The historical leverage and equity volatility of two large mining companies with comparable market exposures is shown in figure 2. Pasminco went into receivership and trading in its shares was suspended in September 2001 (all distress occurring pre-September 11), whereas Rio Tinto remained

1. France Telecom bond spread and volatility



2. Two companies' volatility versus leverage



healthy. The trend expected from theory can be witnessed, though it becomes most apparent around the time when the companies' financial statements are released (indicated by the Leverage data markers in figure 2).

Model

☐ **Basics.** The basic stochastic differential equation for asset price A is:

$$d\ln A = \left(r[t] - \sigma_A{}^2/2\right)dt + \sigma_A dz_A \tag{6}$$

where t is time, σ_A is asset volatility, $r[t]$ is the time-dependent risk-free rate (constant in 'classic' Merton) and dz_A is the risk-neutral Weiner process. This is the basis for most of the remainder of this article, though where 'risk-free' interest rates are also volatile a more general Itô process might then be used for A:

$$d\ln A = \left(r[t] - \sigma_A[t,A,r]^2/2\right)dt + \sigma_A[t,A,r]dz_A \tag{7}$$

combined with, for example, a Black-Derman-Toy (1990) short interest rate process for r:

$$d\ln r = \left(\theta[t] + \sigma_r'[t]/\sigma_r[t]\right)dt + \sigma_r[t]dz_r \tag{8}$$

which is mean-reverting for interest rate volatility declining with regard to time, that is, slope, $\sigma_r'[t]$, is negative, and correlation between A and r, ρ, is:

$$\rho dt = E\left\{dz_A dz_r\right\}, \text{where } E\{\ \} \text{ is the expectation} \tag{9}$$

As an example of the effect of (8) and (9), setting $\sigma_A[t, A, r]$, σ_r and ρ to 40%, 20% and –0.2 for a simple highly leveraged company could result in around a 10-basis-point reduction in theoretical structurally driven credit margin. The converse is true for positive ρ.

It's worth mentioning that alternate processes could be used in place of the geometric Brownian motion described by (6) or (7). The substitution of a jump-diffusion process, for example, may allow for a better fit of the model to the short-term credit margin term structure.

☐ **Valuation framework.** Equation (6) applies in the time between promised cashflows. At times τ let $D[\tau, R]$ be the payment to the creditor ranked R and $A[\tau, R]$ be the asset value after the payment. Then, in the absence of debt covenants:

$$D[\tau,R] = \min\left\{D_F[\tau,R], A[\tau,R+1]\right\} \tag{10}$$

and:

$$A[\tau,R] = A[\tau,R+1] - D[\tau,R] \tag{11}$$

for $R = 0, 1, \ldots, M$, where $R = 0$ is equity, $D[\tau, 0]$ is a dividend, M is the highest ranked liability and $A[\tau, M + 1]$ is the asset value before any payments.

The asset value after all M payments are made is:

$$A[\tau,0] = A[\tau,M+1] - \sum_{R=0}^{M} D[\tau,R] >= 0 \tag{12}$$

Valuation proceeds by 'rolling' the liability values back through time from the time of the longest-dated cashflow. For a Cox-Ross-Rubinstein (1979) binomial tree:

$$L[\tau,R] = e^{-r[t]\times d\tau}\left(p\times L_{UP}[\tau+d\tau,R] - (1-p)\times L_{DOWN}[\tau+d\tau,R]\right) \tag{13}$$

where $d\tau$ is the time step spacing, p is the probability of an up jump, and L_{UP} and L_{DOWN} are the Rth liability values in the up and down states at time $\tau + d\tau$. Note that if bankruptcy had occurred at this point or earlier then the 'rollback' values L_{UP} and L_{DOWN} would be zero because of the requirement to immediately repay all outstanding debt. In the situation where promised payments are contingent upon future liability values, like when dividends are assumed to be proportional to share price, then the simultaneous solution of all these values is required.

☐ **Covenants and other early exercise events that alter capital structure.** Some corporate liabilities may feature early exercise and the capital structure may therefore be contingent upon the 'path' taken by the asset value through time. Then both exercise and non-exercise liability values need to be rolled back to determine if and when early exercise may be optimal (see the convertible bond example below).

A debt covenant is a lender's option to wind up the debt prior to the maturity of the loan. The option can be exercised only if certain balance sheet ratios are breached. A typical covenant could be paraphrased 'debt is called if outstanding principal divided by total assets exceeds 75%'. We model this as a barrier option with a trigger of D/A and payoff P, which is contract-specific but could look like:

$$P[C] = \min\left\{rfv\{C\}, A - Q\right\} \tag{14}$$

where T is time to debt maturity, $rfv\{R\}$ is the risk-free fair value of the Rth ranked debt, C is the rank of the debt with the covenant (typically $C = M$, most senior), and Q is the cost of liquidating the firm. For a coupon bond, $rfv\{C\}$ at time τ may be specified as:

$$rfv\{C\} = F[C]\times\exp\left\{\left(r + m[C]\right)\left(\tau - \tau_p\right)\right\} \tag{15}$$

where $F[C]$ is the debt face value, $m[C]$ is the spread and τ_p is the time of the most recently paid coupon. The barrier is 'outside' in the sense that the trigger is an index, Bermudan because the specified balance-sheet ratios can only be monitored periodically, and is American-style in the sense that a lender will only trigger the covenant if $P[\tau, C] > L[\tau, C]$, where $L[\tau, C]$ is the value of the debt at τ if the covenant is not triggered. Once a senior lender triggers a covenant, lenders more junior are 'locked out' and are assumed to recover their outstanding principal only to the extent that:

$$A[\tau,C] - Q > P[\tau,C] \tag{16}$$

☐ **Debt/equity hybrids.** Debt/equity hybrids, such as convertible bonds, involve optional conversion between debt and equity. Conversion prior to maturity can usually be initiated by the holder, and in some cases also by the issuer. Conversion usually results in the creation of new shares and the simultaneous extinction of the hybrid, resulting in a change in the company's capital structure and dilution of the shareholders' equity. In this model the value of equity at hybrid maturity is calculated for both possible cases, that is, total equity after conversion, $L_C[\tau_m, 0]$, and total equity after non-conversion, $L_N[\tau_m, 0]$, such that:

$$L_C[\tau_m,0] = A[\tau_m,1] - \left(\sum_{R=1}^{M} L[\tau_m,R] - L[\tau_m,CB]\right) \tag{17}$$

where τ_m is the maturity of the hybrid, CB is the rank of the hybrid, $A[\tau_m, 1]$ is the pre-dividend asset value at τ_m,

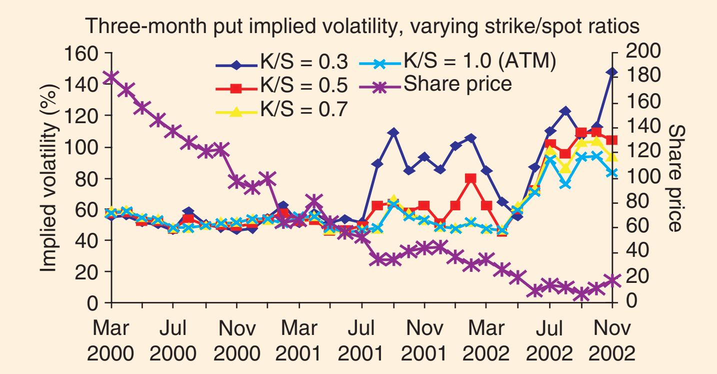
$$\left(\sum_{R=1}^{M} L[\tau_m,R] - L[\tau_m,CB]\right)$$

is the value of the non-equity liabilities excluding the value of the hybrid, and:

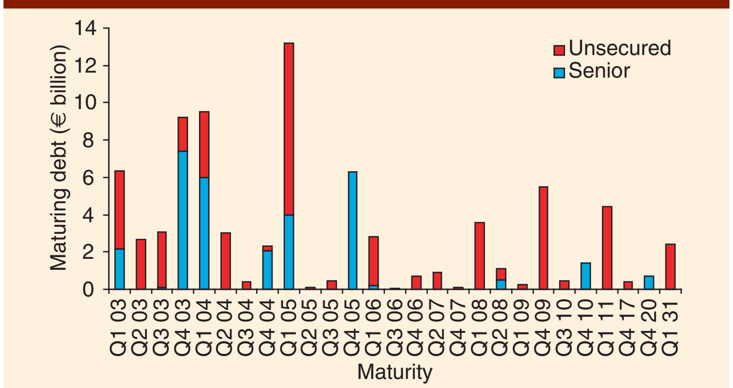
$$L_N[\tau_m,0] = A[\tau_m,1] - \sum_{R=1}^{M} D[\tau_m,R] \tag{18}$$

At times prior to hybrid maturity, both the issuer and the holder need to

3. France Telecom share price and volatility



4. Approximate France Telecom borrowings



be able to 'look ahead' and weigh up the value of converting early against remaining unconverted, so all liability values for both possible outcomes may need to be rolled back during the valuation.

☐ **Corporate actions and policies.** It is possible to model the impact of given board policies, such as dividend payment amounts, debt refinancing or even raising new capital. Perhaps easiest to model is dividend policy, for example, that the board won't pay a dividend if it would lead to bankruptcy, or that dividends are proportional to share price.

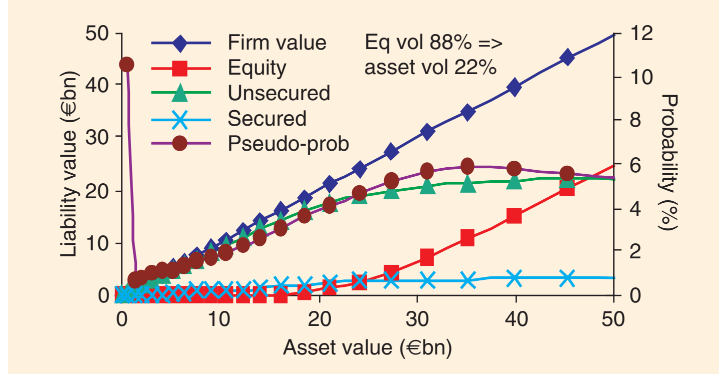
☐ **The problem of 'unobservable' asset value and volatility.** Variants of the Merton model share, to a greater or lesser extent, the problem of determining the value and volatility of the assets.[1] When given equity value and volatility, equation (5) can be a useful first-order approximation for simple capital structures, though as Crosbie (1997) and others have noted it can be quite wrong in circumstances just when you want it to work. The approach that we have taken is to invert the problem. That is, given the current 'market' price and volatility of one of the company's liabilities, we solve for values of asset value and volatility such that the liability's value and volatility, as calculated from the framework's finite differences, match the market's. This would enable debt prices to be used instead of equity if deemed more reliable, and the method can be extended to use volatility smiles as well.

Examples

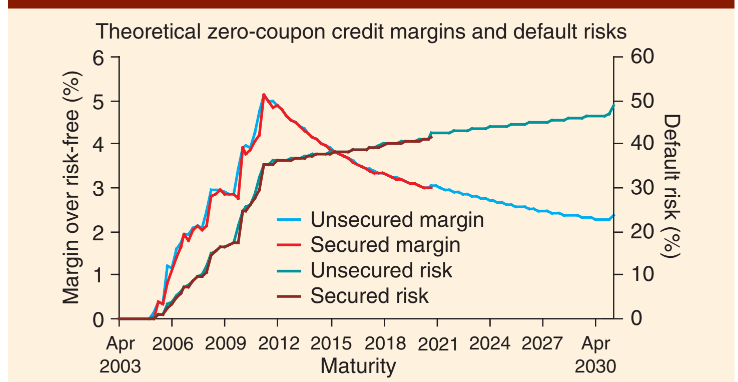
☐ **France Telecom case study: valuation of a complex capital structure.** At December 31, 2002, France Telecom's debt and market capitalisation was approximately €66 billion and €19 billion respectively. In the preceding three years, its share price fell from €219 to a low of less than €7, and its credit rating fell from AA+ to BBB–, with volatility doubling and volatility skews steepening sharply (see figure 3).

France Telecom's debt obligations were a cocktail of fixed and floating secured and unsecured bonds and bank debt. The quarterly debt profile shown in figure 4 is used to generate a time series of promised interest and principal payments. Unsecured liabilities are treated as subordinated.

5. France Telecom five-year liability values



6. France Telecom margins and default risk



Spot equity value and volatility is fed into the above framework implemented as a binomial lattice. Debt covenants were not applied, nor were any adjustments made for the high level of French government ownership.

Figure 5 shows liability values at a 'time slice' through the model grid points at December 2007. Between year one and year five, most secured debt is scheduled for repayment, whereas much of the unsecured debt remains outstanding. Equity behaves like a call option, secured debt behaves like a package of a long risk-free bond and a short put, and the unsecured debt behaves like a bull-spread that, at year five, has an inflection point between €5 billion and €10 billion. Importantly, the year-five pseudo-probability is non-zero (about 10%) at the grid points representing zero asset (and equity) value corresponding to bankruptcy at, or prior to, year five.

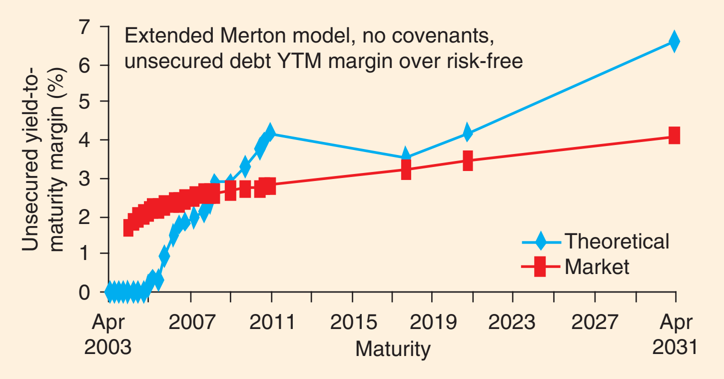
The impact of model-generated bankruptcy is evident in figure 6. The theoretical present value of the promised cashflows of a given priority at any single time can be calculated by assigning just those cashflows to a fictitious intermediate lower priority. Doing this iteratively for each quarter and calculating the implied credit margins gives the margins in figure 6. Also shown is an index of the risk-neutral default risk derived by equating the model's net present value for the promised payment with a risk-weighted value of receiving either all or none of the payment. This index can be a useful way of ranking the relative credit riskiness of a number of companies.

A bond's theoretical yield and hence margin can be calculated from the model's NPVs of its coupons. Figure 7 shows theoretical and market credit margins for large and liquid unsecured bonds. All plotted bonds are euro-denominated except for the 2031 bond, which is dollar-denominated and hence slightly suspect (a euro-denominated bond of roughly the same maturity was issued in early 2003, and it traded at a margin considerably less than that shown here). The model's margin spans the market's. Also evident is the classic Merton model underestimation of short-term credit margin.

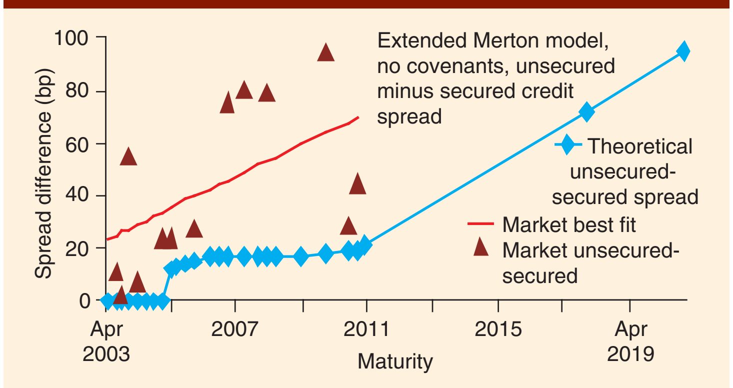
Figure 8 shows the spreads between the secured and unsecured bonds.

[1] *However, Gemmill (2002) studies closed-end funds whose asset prices and historical volatilities are observable in the market*

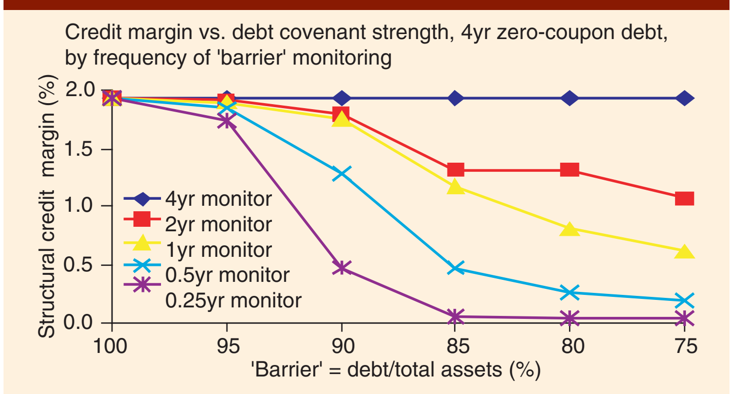
7. France Telecom credit margins



8. France Telecom bond credit spread



9. Debt covenant on simple zero-coupon debt



We have included some illiquid bonds to increase sample size. The market spread is in all cases greater than the theoretical spread, which could be in part due to the omission of debt covenants. The discrepancy is smaller and tighter for shorter-dated bonds and widens to about 60bp past five years.

□ **Debt covenants: effects of 'trigger level' and 'trigger monitoring' frequency.** Debt covenants are modelled as an outside Bermudan up-and-out barrier call option (see above). Here, the model calculates 'ratio of debt to total assets' (RDTA) and monitors it periodically. If, during a monitoring period, RDTA is greater than the trigger level then the debt is called. This has the effect of reducing the 'fair' credit margin required by the lender, as shown for a four-year zero-coupon loan in figure 9. The ratio of debt face value to total assets starts at 70%, the asset's volatility is 30% and the risk-free rate 5%. The strength of the covenant, as indicated by the size of the reduction in required credit margin, clearly increases with increased monitoring frequency and a lower RDTA trigger level. Quarterly monitoring at up to 85% RDTA renders the loan practically risk-free (at least in theory).

□ **Convertible bonds: conventional versus extended Merton.** Convertible bond valuation represents a special challenge because of the interaction between debt and equity, and the role played by credit risk. Conventional convertible bond models have often used lognormal equity price processes and risk-neutral valuation for equity conversion-generated payouts, and an exogenous credit margin for the valuation of coupon and bond redemption-type cashflows. Such models have been found (Ammann, Kind & Wilde, 2003) to generally overvalue convertibles. An extended Merton model, with its in-built credit risk, should have a clear advantage over these models.

■ **Conventional model.** Figure 10 shows a zero-coupon convertible bond valuation adapted from Hull (2002). The bond has nine months to maturity and is convertible into two non-dividend-paying shares at any time. The issuer may attempt to redeem the bond at any time by paying the bondholder $125, though the bondholder could then pre-emptively convert. Nodes D, G and H show states where the share price is high enough that conversion to shares is inevitable, and likewise nodes F, I and J are states where redemption for $100 cash is inevitable. Nodes A, B, C and E show states where either redemption or conversion are possible eventual outcomes. At node E, for example, the discounted expected value of converting into two shares is $63.29, and of remaining unconverted is $55.46 resulting in a net present value at node E of $118.75. Importantly, at node B the discounted expected value of the bond before any actions is $163.10 and the value of converting into two shares is only $152.96, so the bondholder would prefer to remain unconverted. The issuer, however, can buy the bond back for $125 and so it calls the bond. This then activates the holder's option to immediately convert and take the $152.96. The resultant valuation is $120.18.

The conventional model suffers from a number of shortcomings. These include: the inability to explicitly model the bankruptcy that is implied by the high credit margin of 5%, which means extreme volatility skews and an arbitrarily defined credit margin are common in practice; and capital restructuring such as dilution upon conversion is difficult to capture.

■ **Extended Merton model.** Figure 11 illustrates the same convertible bond in our extended Merton framework. We've postulated some issuer details not required by the conventional model, such as the capital structure of the company and the number of shares and convertibles on issue. A senior debt of $150 million face value explains the 5% credit margin on the convertible bond. The underlying random variable is the issuer's asset value, specified such that the equity, a call, is worth $50 and has 85% volatility as before. The convertible is an exchange option involving defaultable subordinated debt and shareholder dilution. Observe that in node I, the holder would redeem their bond for the full face value, $20 million/200,000 = $100, but that in node J the issuer's assets will only cover a $2.33 redemption. A call would cost the issuer $125 × 200,000 = $25 million, so again the issuer calls the bond at node B.

The extended Merton valuation is thus $22.38 million/200,000 = $111.90 per bond, or about 7% less than the value given by the conventional model.

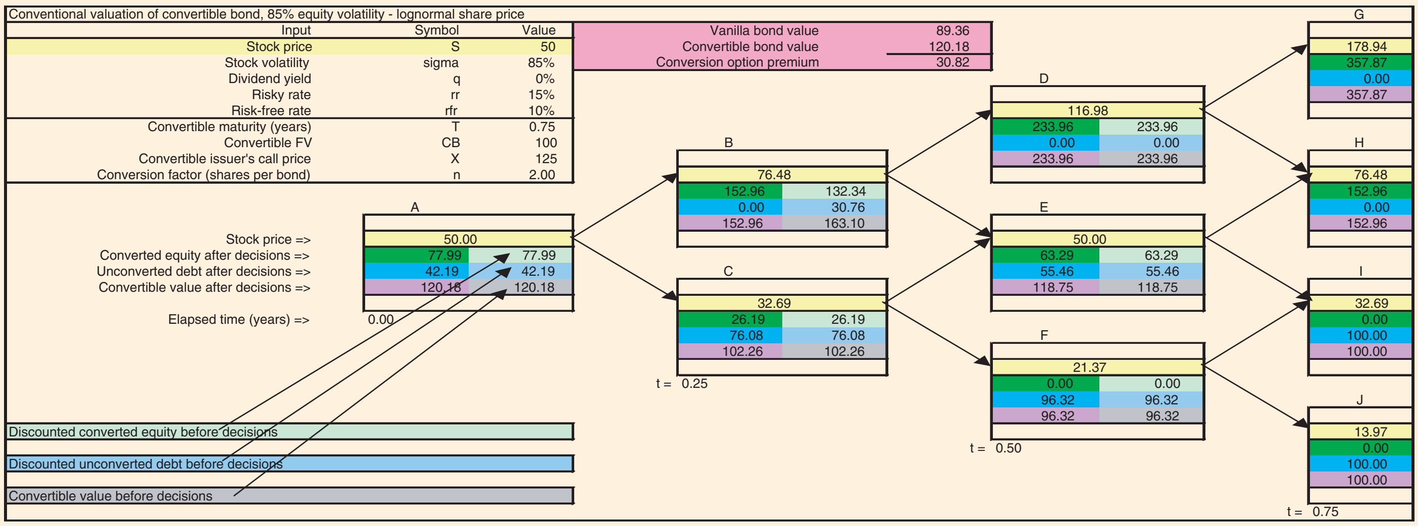
Conclusion

We have described a framework based on a structural model for the relative valuation of all the liabilities (for example, debt, equity, convertibles, hybrids) issued by a single limited liability firm. The proposed extensions to Merton's original model provide for significantly more realistic valuation scenarios, as demonstrated by the France Telecom case study and the convertible bond models comparisons.

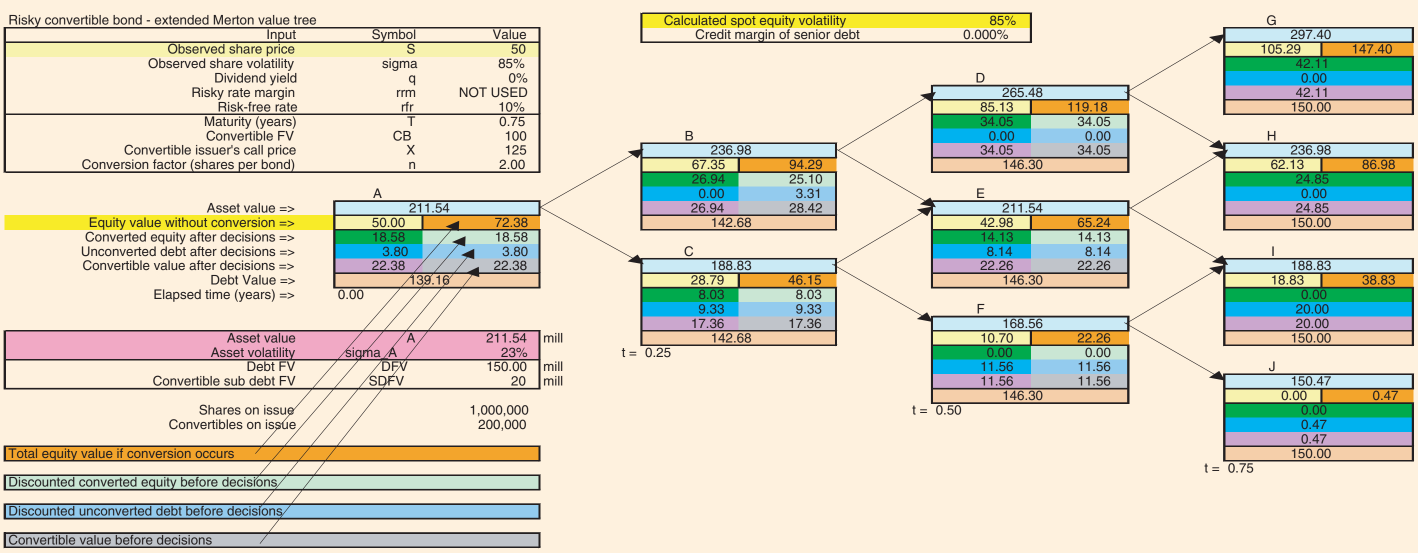
We believe the above framework can provide substantial insights into so-called 'relative value trading' or 'capital structure arbitrage', though there are many areas for possible improvement in both the specification of the model (for example, asset value with jump-diffusion, additional corporate actions policy definitions), and also in the implementation of the framework (for example, a more robust partial differential equation implementation, or Monte Carlo simulation with early exercise features). ■

Doug Moss is with the quantitative applications division at Macquarie Bank in Sydney. Email: Doug.Moss@macquarie.com

10. Example of a conventional convertible bond valuation



11. Example of an extended Merton convertible bond valuation



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