

There are an Awful Lot of Options in Brazil

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ABSTRACT. Brazil, faced with financial problems different from other economies, has evolved instruments to cater for its needs that are not seen elsewhere. Interestingly, its market has also priced one option in a way that implies an underlying BGM model.

CONTENTS

1	Introduction	1
2	Pre-DI Swaps and Swaptions	2
3	DI Index Options	5
4	DI Futures Contracts	6
5	DI Futures Options	8
A	Miscellaneous Formulae	8

1. INTRODUCTION

At the outset, the author would like to emphasize that this note, which addresses some of the theoretical aspects of Brazilian options, is work in progress and may contain mistakes, biased interpretations or simply reflect his misunderstanding of actual practice (details of some instruments are a struggle, in particular, the system of contract dates). For more information, readers might like to refer to the interest rate section of the BMF (*Bolsa de Mercadorias & Futuros*) English language website at <http://www.bmf.com.br/IndexEnglish.asp>.

Without ever having lived in Brazil, the author nevertheless imagines that frequent financial crises will naturally lead to the invention of safe and flexible ways of increasing (or at least maintaining) wealth over short time horizons. For that reason he is not surprised that the foundation for most Brazilian interest rate derivatives is the **CDI rate** or *overnight DI (Deposito Interbancario) rate* $D(t)$, which is an annualized rate paying

$$[1 + D(t)]^{\frac{1}{252}}$$

over a one day period at time t . It is calculated and published daily by ANDIMA and represents the average rate of all inter-bank overnight transactions in Brazil: banks usually express their cost of funding as a percentage of the published CDI terms.

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The **IDI index** accumulates those daily payments from some start date

$$\text{IDI}(t) = \prod_{i=0}^{n-1} [1 + D(t_i)]^{\frac{1}{252}}, \quad n = \lfloor 252 \times t \rfloor$$

(resetting to 100,000 from time-to-time). The IDI index therefore behaves like a bank account, and so it can be reasonably modelled in HJM style by setting

$$\begin{aligned} [1 + D(t)]^{\frac{1}{252}} &= 1 + r(t) \Delta(t) \cong \exp r(t) \Delta(t), \\ \text{IDI}(t) &= \beta(t) = \exp \int_0^t r(s) ds \end{aligned} \quad (1)$$

(where $\Delta(t)$ is one day and $r(t)$ is the spot interest rate) or, in spot Libor fashion, by rolling it up into consecutive zero coupon bonds that mature daily.

2. PRE-DI SWAPS AND SWAPTIONS

These are standard over-the-counter deals tailored to the needs of the counterparties, with the convention (the author's understanding) that the length of swaps is always a whole number of months (that is, deals begin and end on the same day of the month).

By definition the **Pre DI payer swap** struck at K and accumulated over the interval $[T, T_1]$, where $T_1 = \delta + T$, pays

$$\frac{\text{IDI}(T_1)}{\text{IDI}(T)} - K = \frac{\beta(T_1)}{\beta(T)} - K$$

at time T_1 . The time t **swaprte** $\omega(t) = \omega(t, T, T_1)$ is that value of K which makes the swap's time t value zero

$$\mathbf{E}_0 \left\{ \frac{1}{\beta(T_1)} \left[\frac{\beta(T_1)}{\beta(T)} - \omega(t) \right] \middle| \mathcal{F}_t \right\} = 0,$$

namely

$$\omega(t) = \frac{B(t, T)}{B(t, T_1)} = \frac{1}{F_T(t, T_1)},$$

where \mathbf{E}_0 is expectation under the spot measure \mathbb{P}_0 , $B(t, T)$ is the time t value of a zero coupon bond maturing at time T , and $F_T(t, T_1)$ is a T -forward contract on the zero maturing at T_1 .

To make numbers comparable with the CDI rate, the convention is that the actual market **quoted swaprte** $f(t, T, T_1)$ and actual market **quoted strike** κ are obtained from $\omega(t, T, T_1)$ and K by the unique one-to-one formulae

$$\omega(t) = \omega(t, T, T_1) = [1 + f(t, T, T_1)]^{\frac{T_1 - T}{252}}, \quad K = [1 + \kappa]^{\frac{T_1 - T}{252}}. \quad (2)$$

Also, to develop models, we will need the notion of **forward accrual** $A(t)$ over the interval $[T, T_1]$

$$A(t) = A(t, T, T_1) = [1 + f(t, T, T_1)]^{\frac{T_1 - T}{252}} - 1 = \omega(t) - 1 = \frac{1}{F_T(t, T_1)} - 1, \quad (3)$$

which is virtually (give or take a scaling factor) the simple forward over $[T, T_1]$.

Substituting back, the time t value of a payer **swap** (pay fixed receive floating) is therefore

$$\begin{aligned} \text{pswap}(t) &= \text{pswap}(t, T, T_1) & (4) \\ &= \mathbf{E}_0 \left\{ \frac{1}{\beta(T_1)} \left[\frac{\beta(T_1)}{\beta(T)} - K \right] \middle| \mathcal{F}_t \right\} = \mathbf{E}_0 \left\{ \frac{1}{\beta(T_1)} [\omega(t) - K] \middle| \mathcal{F}_t \right\}, \\ &= B(t, T_1) [\omega(t) - K] = B(t, T) - K B(t, T_1), \\ &= B(t, T_1) [A(t) - (K - 1)], \end{aligned}$$

which has exactly the same form as any standard swap, except that there is just one exchange.

The time t value of the corresponding payer **swaption** can be expressed in several ways

$$\begin{aligned} \text{pswpn}(t) &= \text{pswpn}(t, T, T_1) & (5) \\ &= \beta(t) \mathbf{E}_0 \left\{ \frac{B(T, T_1)}{\beta(T)} [\omega(T) - K]^+ \middle| \mathcal{F}_t \right\} \\ &= B(t, T) \mathbf{E}_T \left\{ [1 - KF_T(T, T_1)]^+ \middle| \mathcal{F}_t \right\}, \\ &= B(t, T_1) \mathbf{E}_{T_1} \left\{ [A(T) - (K - 1)]^+ \middle| \mathcal{F}_t \right\}, \end{aligned}$$

where \mathbf{E}_T (respectively \mathbf{E}_{T_1}) is expectation under the T -forward measure \mathbb{P}_T (respectively \mathbb{P}_{T_1}). Similar expressions hold for receivers

$$\begin{aligned} \text{rswap}(t) &= \text{rswap}(t, T, T_1) = K B(t, T_1) - B(t, T), & (6) \\ &= B(t, T_1) [(K - 1) - A(t)], \\ \text{rswpn}(t) &= \text{rswpn}(t, T, T_1) = B(t, T) \mathbf{E}_T \left\{ [KF_T(T, T_1) - 1]^+ \middle| \mathcal{F}_t \right\}, \\ &= B(t, T_1) \mathbf{E}_{T_1} \left\{ [(K - 1) - A(T)]^+ \middle| \mathcal{F}_t \right\}. \end{aligned}$$

Note that these results are **model independent**, and that the options can be regarded as either caps/floors or swaptions because they are on swaps with just one exchange.

2.1. In the HJM framework. An SDE for the forward $F_T(t, T_1)$ under \mathbb{P}_T is

$$\begin{aligned} dF_T(t, T_1) &= -F_T(t, T_1) \int_T^{T_1} \sigma^*(t, u) du dW_T(t) \Rightarrow \\ F_T(T, T_1) &= F_T(t, T_1) \mathcal{E} \left\{ \int_t^T \int_T^{T_1} \sigma^*(s, u) du dW_T(s) \right\}, \end{aligned}$$

giving, from (5), (6) and (2), Black-Scholes formulae for the swaptions

$$\begin{aligned} \text{pswpn}(t) &= B(t, T) \text{BSput}(K F_T(t, T_1), 1, \zeta^2), \\ \text{rswpn}(t) &= B(t, T) \text{BScall}(K F_T(t, T_1), 1, \zeta^2), \\ \zeta^2 &= \int_t^T \left| \int_T^{T_1} \sigma(s, u) du \right|^2 ds. \end{aligned}$$

In the flat Ho&Lee case that produces a Black option formula with implied volatility $\delta\sigma$.

2.2. In the BGM framework. The SDE for the reciprocal of the forward under \mathbb{P}_{T_1} yields an SDE for the accrual

$$\begin{aligned} d\left(\frac{1}{F_T(t, T_1)}\right) &= \left(\frac{1}{F_T(t, T_1)}\right) \int_T^{T_1} \sigma^*(t, u) du dW_{T_1}(t) \Rightarrow \\ \frac{dA(t, T, T_1)}{A(t, T, T_1)} &= \gamma^*(t, T, T_1) dW_{T_1}(t), \end{aligned}$$

if, in BGM style, the stochastic HJM volatility is chosen to satisfy

$$(1 + A(t, T, T_1)) \int_T^{T_1} \sigma(t, u) du = A(t, T, T_1) \gamma(t, T, T_1),$$

where $\gamma(t, T, T_1)$ is deterministic. From (5), (6) and (2), that gives the following BS formulae for the swaptions

$$\begin{aligned} \text{pswpn}(t) &= \text{pswpn}(t, T, T_1) = B(t, T_1) \text{BScall}(A(t, T, T_1), K - 1, \zeta^2), \\ \text{rswpn}(t) &= \text{rswpn}(t, T, T_1) = B(t, T_1) \text{BSput}(A(t, T, T_1), K - 1, \zeta^2), \\ \zeta^2 &= \int_t^T |\gamma(s, T, T_1)|^2 ds. \end{aligned} \tag{7}$$

3. DI INDEX OPTIONS

These are an exchange traded options, maturing at the beginning of the months Jan, April, July, Oct and also the month following the current month. But as we will see, the volatilities of these options steadily decrease (to zero at maturity), which seems to make them difficult to use in practice; consequently they are neither heavily traded nor used to hedge swaptions.

The payoff at time T for a DI Index option is the accumulated index starting from some reference time which (with no loss of generality) we may take as 0. Hence DI call and put options struck at K will respectively have time t value

$$\begin{aligned} \text{DIcall}(t, T) &= \beta(t) \mathbf{E}_0 \left\{ \frac{1}{\beta(T)} [\beta(T) - K]^+ \middle| \mathcal{F}_t \right\}, \\ &= \mathbf{E}_0 \left\{ \left[\beta(t) - K \exp \left(- \int_t^T r(s) ds \right) \right]^+ \middle| \mathcal{F}_t \right\} \\ \text{DIput}(t, T) &= \mathbf{E}_0 \left\{ \left[K \exp \left(- \int_t^T r(s) ds \right) \right]^+ - \beta(t) \middle| \mathcal{F}_t \right\} \end{aligned}$$

Note that the reference time effectively scales the contract because a later reference time T^* would result in an option payoff of $\frac{\beta(T)}{\beta(T^*)}$ instead of $\beta(T)$.

Getting an option formula is easy in the HJM case (but messy in the BGM framework).

3.1. In the HJM framework. An SDE for the zero coupon bond $B(t, T)$ is

$$dB(t, T) = B(t, T) \left[r(t) dt - \int_t^T \sigma^*(t, u) du dW_0(t) \right], \quad (8)$$

which has time T solution

$$\begin{aligned} B(T, T) = 1 &= B(t, T) \exp \left(\int_t^T r(s) ds \right) \mathcal{E} \left(- \int_t^T \int_s^T \sigma^*(s, u) du dW_0(s) \right), \quad (9) \\ \Rightarrow \exp \left(- \int_t^T r(s) ds \right) &= B(t, T) \mathcal{E} \left(- \int_t^T \int_s^T \sigma^*(s, u) du dW_0(s) \right). \end{aligned}$$

Hence, from (1), the time t values of the options are

$$\begin{aligned} \text{DIcall}(t, T) &= \mathbf{E}_0 \left\{ \left[\beta(t) - K B(t, T) \mathcal{E} \left(- \int_t^T \int_s^T \sigma^*(s, u) du dW_0(s) \right) \right]^+ \middle| \mathcal{F}_t \right\}, \\ &= \text{BSput}(K B(t, T), \beta(t), \zeta^2) \\ \text{DIput}(t, T) &= \text{BScall}(K B(t, T), \beta(t), \zeta^2) \\ \zeta^2 &= \int_t^T \left| \int_s^T \sigma(s, u) du \right|^2 ds. \end{aligned}$$

which are Black-Scholes expressions with volatilities contracting to zero. In the flat **Ho&Lee** case they yield the corresponding BS implied volatility

$$\frac{1}{\sqrt{3}}(T - t)\sigma.$$

4. DI FUTURES CONTRACTS

Futures contracts and options on them, are exchange traded instruments that mature at the beginning of the months Jan, April, July, Oct and also the month following the current month; hence underlying contracts may have 1,3,6 or 12 months to run. The options are often used to hedge OTC swaptions, because their dynamics are similar (give or take differences in maturity) and their volatilities well behaved.

The time T maturing futures contract can be entered or exited at any time without cost, its numerical *value* equals that of a zero coupon bond $B(t, T)$ maturing at the same time, and the *daily margin payments* $\Delta M_T(t)$ associated with it are, from (1) with Δt one day,

$$\begin{aligned} \Delta M_T(t) &= B(t + \Delta t, T) - B(t, T) [1 + D(t)]^{\frac{1}{252}}, \\ &= B(t + \Delta t, T) - B(t, T) [1 + r(t) \Delta t]. \end{aligned}$$

In the continuous case the dynamics (8) of a zero coupon contract imply

$$dM_T(t) = dB(t, T) - B(t, T) r(t) dt = -B(t, T) \int_t^T \sigma^*(t, u) du dW_0(t), \quad (10)$$

that is, the daily margin payment equals the daily change in the stochastic part of the zero coupon bond. Hence the time t value of a DI futures contract (exiting at an arbitrary time $T_1 \leq T$) being equal to the present value of the margin payments

$$\mathbf{E}_0 \left\{ \int_t^{T_1} \frac{dM_T(s)}{\beta(s)} \middle| \mathcal{F}_t \right\} = -\mathbf{E}_0 \left\{ \int_t^{T_1} \frac{B(s, T)}{\beta(s)} \int_s^T \sigma^*(s, u) du dW_0(s) \middle| \mathcal{F}_t \right\} = 0,$$

must be zero. This must be true, of course, because of the zero cost of entering and leaving the contract.

Remark 1. *Although the DI contract delivers nothing (in contrast to a standard futures contract which delivers something even if cash settled), it is a futures contract in the sense that it costs nothing to enter or leave and settles at the margin. Also the margin payments can be duplicated by borrowing to purchase a zero coupon and financing it at the overnight rate.*

An **alternative approach** is to take the given properties of the contract and posit an index $B(t, T)$ which:

1. Converges to unity at its *maturity* T , that is $B(T, T) = 1$,
2. Can be entered and exited freely with margin payments

$$dM_T(t) = dB(t, T) - B(t, T) r(t) dt$$

that have zero present value.

Using an argument similar to the one above, for the margin payments to have zero value whatever entry and exit times clearly $M_T(t)$ is a \mathbb{P}_0 martingale, and so $B(t, T)$ satisfies an SDE like

$$\frac{dB(t, T)}{B(t, T)} = r(t) dt + \xi(t, T) dW_0(t)$$

for some well behaved volatility function $\xi(t, T)$. The solution at maturity T

$$\begin{aligned} B(T, T) = 1 &= B(t, T) \exp\left(\int_t^T r(s) ds\right) \mathcal{E}\left(-\int_t^T \xi(s, T) dW_0(s)\right), \\ \Rightarrow B(t, T) \mathcal{E}\left(-\int_t^T \xi(s, T) dW_0(s)\right) &= \exp\left(-\int_t^T r(s) ds\right), \\ \Rightarrow B(t, T) &= \mathbf{E}_0\left\{\exp\left(-\int_t^T r(s) ds\right) \middle| \mathcal{F}_t\right\}, \end{aligned}$$

which is a zero coupon bond.

4.1. Hedging with futures contracts. The futures contract can be used to directly hedge pre-DI swaps, because from (4), (8) and (10)

$$d\text{pswap}(t) = \text{pswap}(t) r(t) dt + dM_T(t) - K dM_{T_1}(t),$$

and this result is **independent of the model** (no assumptions made about the HJM volatility which may be stochastic).

In the HJM framework, using (3) we can hedge swaptions via

$$d\text{pswpn}(t) = \text{pswpn}(t) r(t) dt + \text{BSput}(K F_T(t, T_1), 1, \zeta^2) dM_T(t) \\ + \Delta \text{BSput}(K F_T(t, T_1), 1, \zeta^2) [dM_{T_1}(t) - F_T(t, T_1) dM_T(t)],$$

and DI Index options via

$$d\text{DIcall}(t, T) = \text{DIcall}(t, T) r(t) dt + \Delta \text{BSput}(K B(t, T), \beta(t), \zeta^2) K dM_T(t).$$

Similarly **in the BGM framework** we can hedge swaptions via

$$d\text{pswpn}(t) = \text{pswpn}(t) r(t) dt + \text{BScall}(A(t, T, T_1), K - 1, \zeta^2) dM_{T_1}(t) \quad (11) \\ + \Delta \text{BScall}(A(t, T, T_1), K - 1, \zeta^2) [dM_T(t) - (1 + A(t, T, T_1)) dM_{T_1}(t)].$$

5. DI FUTURES OPTIONS

This is an option on a zero coupon bond maturing at time T_1 with payoffs at time T like

$$\text{payer}(T) = \left[B(T, T_1) - \frac{1}{[1+\kappa]^{\frac{T_1-T}{252}}} \right]^+ = \left[B(T, T_1) - \frac{1}{K} \right]^+, \\ \text{receiver}(T) = \left[\frac{1}{[1+\kappa]^{\frac{T_1-T}{252}}} - B(T, T_1) \right]^+ = \left[\frac{1}{K} - B(T, T_1) \right]^+,$$

where $K = [1 + \kappa]^{\frac{T_1-T}{252}}$. Hence from (3)

$$\text{payer}(t) = B(t, T_1) \mathbf{E}_{T_1} \left\{ \frac{1}{B(T, T_1)} \left[B(T, T_1) - \frac{1}{K} \right]^+ \middle| \mathcal{F}_t \right\}, \\ = \frac{1}{K} B(t, T_1) \mathbf{E}_{T_1} \{ [(K - 1) - A(T)]^+ | \mathcal{F}_t \}, \\ = \frac{1}{K} B(t, T_1) \text{BScall}(A(t, T, T_1), K - 1, \zeta^2). \\ \text{receiver}(t) = \frac{1}{K} B(t, T_1) \text{BSput}(A(t, T, T_1), K - 1, \zeta^2).$$

These formulae are almost the same as those for swaptions (7), and so hedges as similar to those given by (11).

A. MISCELLANEOUS FORMULAE

1. Let X and Y be jointly normally distributed with zero mean, and set $\zeta^2 = \text{var}(X - Y)$. Then

$$C = \mathbf{E}[K\mathbf{e}(X) - L\mathbf{e}(Y)]^+ = K\mathbf{N}(h) - L\mathbf{N}(h - \zeta), \quad \text{where} \\ h = \frac{\ln\left(\frac{K}{L}\right) + \frac{1}{2}\zeta^2}{\zeta}, \quad \mathbf{e}(X) = \exp\left(X - \frac{1}{2}\text{var} X\right),$$

moreover, Greeks are given by

$$\begin{aligned} \text{Delta } \Delta \quad \partial_K C &= \mathbf{N}(h), \quad \partial_L C = -\mathbf{N}(h - \zeta), \\ \text{Vega } \Lambda \quad \partial_\zeta C &= K\mathbf{N}'(h) = L\mathbf{N}'(h - \zeta), \\ \text{Gamma } \Gamma \quad \partial_K^2 C &= \frac{1}{K\zeta}\mathbf{N}'(h), \quad \partial_L^2 C = \frac{1}{L\zeta}\mathbf{N}'(h - \zeta), \\ \partial_K \partial_L C &= -\frac{1}{L\zeta}\mathbf{N}'(h) = -\frac{1}{K\zeta}\mathbf{N}'(h - \zeta). \end{aligned}$$

2. From the above

$$\begin{aligned} \text{BScall}(S, \kappa, \zeta^2) &= \mathbf{E}[S\mathbf{e}(X) - \kappa]^+ = S\mathbf{N}(h) - \kappa\mathbf{N}(h - \zeta) \\ \text{BSput}(S, \kappa, \zeta^2) &= \mathbf{E}[\kappa - S\mathbf{e}(X)]^+ = \kappa\mathbf{N}(-(h - \zeta)) - S\mathbf{N}(-h) \\ h &= \frac{\ln\left(\frac{S}{\kappa}\right) + \frac{1}{2}\zeta^2}{\zeta}, \quad \zeta^2 = \text{var } X \end{aligned}$$

with deltas

$$\begin{aligned} \Delta \text{BScall}(S, \kappa, \zeta^2) &= \partial_S \text{BScall}(S, \kappa, \zeta^2) = \mathbf{N}(h), \\ \Delta \text{BSput}(S, \kappa, \zeta^2) &= \partial_S \text{BSput}(S, \kappa, \zeta^2) = -\mathbf{N}(-h). \end{aligned}$$

3. Let $f(t, X_t)$ be an arbitrage free instrument in which the driver X_t is a diffusion with SDE

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_0(t)$$

under the spot measure \mathbb{P}_0 . Because $\beta(t) = \exp\left(\int_0^t r(s) ds\right)$ is a finite variation process, $\frac{f(t, X_t)}{\beta(t)}$ is a \mathbb{P}_0 martingale, and stochastic and drift parts on both sides of an equation must match

$$d\left\{\frac{f(t, X_t)}{\beta(t)}\right\} = (\text{expression}) dW_0(t) = \frac{1}{\beta(t)} \frac{\partial}{\partial x} f(t, X_t) \sigma(t, X_t) dW_0(t).$$

Hence the following SDE for $f(t, X_t)$

$$df(t, X_t) = f(t, X_t) r(t) dt + \frac{\partial}{\partial x} f(t, X_t) \sigma(t, X_t) dW_0(t).$$