Introduction

BMF (*Bolsa de Mercadorias & Futuros*) English language website at <http://www.bmf.com.br/IndexEnglish.asp>

Foundation for interest rate intruments is **CDI rate** *rate* $D(t)$, an annualized overnight rate paying

$$[1 + D(t)]^{\frac{1}{252}}$$

CDI rate is average of all inter-bank overnight transaction rates in Brazil

Banks usually express their cost of funding as a percentage of the CDI rate

The **IDI index** accumulates the CDI rate

$$\mathrm{IDI}(t) = \prod_{i=0}^{n-1} [1 + D(t_i)]^{\frac{1}{252}}, \qquad n = \lfloor 252 \times t \rfloor$$

IDI like HJM bank account so model like

$$[1 + D(t)]^{\frac{1}{252}} = 1 + r(t)\Delta(t) \cong \exp r(t)\Delta(t)$$

$$\mathrm{IDI}(t) = \beta(t) = \exp \int_0^t r(s)ds$$

Pre-DI Swaps/Swaptions

OTC, length swaps whole # months

Pre_DI payer swap **strike** K over $[T, T_1]$ (where $T_1 = T + \delta$) pays at T_1

$$\frac{\mathrm{IDI}(T_1)}{\mathrm{IDI}(T)} - K = \frac{\beta(T_1)}{\beta(T)} - K$$

Swaprate $\omega(t) = \omega(t, T, T_1)$ that value K which makes swap value zero

$$\mathbf{E}_0 \left\{ \frac{1}{\beta(T_1)} \left[\frac{\beta(T_1)}{\beta(T)} - \omega(t) \right] \middle| \mathsf{F}_t \right\} = 0$$

$$\Rightarrow \quad \omega(t) = \frac{B(t,T)}{B(t,T_1)} = \frac{1}{F_T(t,T_1)}$$

To make numbers comparable to CDI **quoted** swaprate $f(t, T, T_1)$ strike κ are

$$\omega(t) = \omega(t, T, T_1) = [1 + f(t, T, T_1)]^{\frac{T_1 - T}{252}}$$

$$K = [1 + \kappa]^{\frac{T_1 - T}{252}}.$$

To model need notion **forward accrual**

$$A(t) = A(t, T, T_1) = [1 + f(t, T, T_1)]^{\frac{T_1 - T}{252}} - 1$$

$$= \omega(t) - 1 = \frac{1}{F_T(t, T_1)} - 1$$

Hence values payer **swap** & **swaption**

$$\text{pswap}(t) = \text{pswap}(t, T, T_1)$$

$$= \mathbf{E}_0\left\{ \frac{1}{\beta(T_1)} \left[\frac{\beta(T_1)}{\beta(T)} - K \right] \middle| \mathsf{F}_t \right\}$$

$$= \mathbf{E}_0\left\{ \frac{1}{\beta(T_1)} [\omega(t) - K] \middle| \mathsf{F}_t \right\}$$

$$= B(t, T_1)[\omega(t) - K] = B(t, T) - K\, B(t, T_1)$$

$$= B(t, T_1)[A(t) - (K - 1)]$$

$$\text{pswpn}(t) = \text{pswpn}(t, T, T_1)$$

$$= \beta(t)\mathbf{E}_0\left\{ \frac{B(T, T_1)}{\beta(T)} [\omega(T) - K]^+ \middle| \mathsf{F}_t \right\}$$

$$= B(t, T)\mathbf{E}_T\{[1 - KF_T(T, T_1)]^+ | \mathsf{F}_t\}$$

$$= B(t, T_1)\mathbf{E}_{T_1}\{[A(T) - (K - 1)]^+ | \mathsf{F}_t\}$$

Results **model independent** & options either caps/floors or swaptions

In **HJM framework** SDE for $F_T(t, T_1)$

$$dF_T(t, T_1) = -F_T(t, T_1) \int_T^{T_1} \sigma^*(t, u) du\, dW_T(t)$$

$$F_T(T, T_1) = F_T(t, T_1)\,\mathbf{Dex}\left\{ \int_t^T \int_T^{T_1} \sigma^* du\, dW_T(s) \right\}$$

$$\text{pswpn}(t) = B(t,T)\,\text{BSput}(K\,F_T(t,T_1),\ 1,\ \zeta^2)$$

$$\zeta^2 = \int_t^T \left| \int_T^{T_1} \sigma(s,u)du \right|^2 ds$$

For flat Ho&Lee BS implied volatility is $\delta\sigma$

In **BGM framework** SDE reciprocal

$$d\left(\frac{1}{F_T(t,T_1)} \right) = \left(\frac{1}{F_T(t,T_1)} \right) \int_T^{T_1} \sigma du\, dW_{T_1}(t)$$

$$\frac{dA(t,T,T_1)}{A(t,T,T_1)} = \gamma^*(t,T,T_1)\, dW_{T_1}(t)$$

if stoch HJM volatility chosen satisfy

$$(1 + A(t,T,T_1)) \int_T^{T_1} \sigma(t,u)du = A(t,T,T_1)\gamma(t,T,T_1)$$

with $\gamma(t,T,T_1)$ deterministic

Gives BS formulae for swaptions

$$\text{pswpn}(t) = \text{pswpn}(t,T,T_1)$$

$$= B(t,T_1)\,\text{BScall}(A(t,T,T_1), K-1, \zeta^2)$$

$$\zeta^2 = \int_t^T |\gamma(s,T,T_1)|^2 ds$$

DI Index Options

Exchange traded maturing Jan, April, July, Oct & month after current one

Volatilities decrease (to zero at maturity) difficult use neither heavily traded nor used hedge swaptions

The payoff accumulated index from reference time say 0

Call option struck at K

$$\mathrm{DIcall}(t,T) = \beta(t)\mathbf{E}_0\left\{\frac{1}{\beta(T)}[\beta(T)-K]^+\middle|\mathsf{F}_t\right\}$$

$$= \mathbf{E}_0\left\{\left[\beta(t) - K\exp\left(-\int_t^T r(s)ds\right)\right]^+\middle|\mathsf{F}_t\right\}$$

Reference time scales contract cos later reference T^* payoff $\frac{\beta(T)}{\beta(T^*)}$

In the HJM framework SDE zero coupon bond $B(t,T)$

$$dB(t,T) = B(t,T)\left[r(t)dt - \int_t^T \sigma^*(t,u)du\, dW_0(t)\right]$$

which has time T solution

$$B(T,T) = 1$$

$$= B(t,T)\exp\left(\int_t^T r(s)ds\right)\mathbf{Dex}\left(-\int_t^T\int_s^T \sigma du\, dW_0(s)\right)$$

and so $\exp\left(-\int_t^T r(s)ds\right)$

$$= B(t,T)\mathbf{Dex}\left(-\int_t^T\int_s^T \sigma(s,u)du\, dW_0(s)\right)$$

Hence value DI call option

$$\text{DIcall}(t,T)$$

$$= \mathbf{E}_0\left\{\left[\begin{matrix}\beta(t) - K\,B(t,T) \\ \mathbf{Dex}\left(-\int_t^T\int_s^T \sigma du\, dW_0(s)\right)\end{matrix}\right]^+ \middle| \mathsf{F}_t\right\}$$

$$= \text{BSput}(K\,B(t,T),\ \beta(t),\ \zeta^2)$$

$$\zeta^2 = \int_t^T \left|\int_s^T \sigma(s,u)du\right|^2 ds$$

Black-Scholes expressions volatilities contracting 0 flat **Ho&Lee** BS implied volatility

$$\frac{1}{\sqrt{3}}(T-t)\sigma$$

DI Futures Contracts

Futures contracts & options on them exchange traded maturing in Jan, April, July, Oct & also month following current on, hence underlying contracts possible 1,3,6 or 12 months run

Options used hedge OTC swaptions ’cos dynamics similar (apart maturity differences) $+$ volatilities behaved

Time T maturing futures contract can enter/exit any time no cost

Numerical *value* $=$ zero coupon bond $B(t,T)$

Associated **daily margin** payments

$$\begin{aligned}\Delta M_T(t) &= B(t+\Delta t,T) - B(t,T)[1+D(t)]^{\frac{1}{252}} \\ &= B(t+\Delta t,T) - B(t,T)[1+r(t)\Delta t]\end{aligned}$$

Continuous case dynamics zero coupon imply

$$\begin{aligned}dM_T(t) &= dB(t,T) - B(t,T)r(t)dt \\ &= -B(t,T)\int_t^T \sigma^*(t,u)du\; dW_0(t)\end{aligned}$$

So daily margin payment $=$ daily change in stochastic part zero coupon

Hence value DI futures contract (exiting $T_1 \leq T$) $=$ present value margin payments

$$\mathbf{E}_0\left\{\int_t^{T_1} \frac{dM_T(s)}{\beta(s)} \middle| \mathsf{F}_t\right\}$$

$$= -\mathbf{E}_0\left\{\int_t^{T_1} \frac{B(s,T)}{\beta(s)} \int_s^T \sigma^*(s,u)du \, dW_0(s) \middle| \mathsf{F}_t\right\} =$$

must be zero

Must be true 'cos zero cost enter/exit contract

Alternative approach posit index $B(t,T)$ such that

1. $B(T,T) = 1$,
2. Enter/Exit no cost & margin payments

   $$dM_T(t) = dB(t,T) - B(t,T)r(t)dt$$

   have zero present value

For margin payments to have zero value whatever the entry time t or exit time T_1

$$\mathbf{E}_0\left\{\int_t^{T_1} \frac{dM_T(s)}{\beta(s)} \,\middle|\, \mathrm{F}_t\right\} = 0 \quad \Rightarrow$$

$$\mathbf{E}_0\left\{\int_0^{T_1} \frac{dM_T(s)}{\beta(s)} \,\middle|\, \mathrm{F}_t\right\} = \int_0^t \frac{dM_T(s)}{\beta(s)}$$

$dZ(t) = \dfrac{dM_T(t)}{\beta(t)}$ is P_0 martingale $\Rightarrow$

$dM_T(t) = \beta(t)dZ(t)$ is P_0 martingale

Hence $B(t,T)$ satisfies an SDE like

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + \xi(t,T)dW_0(t)$$

with solution at T

$$B(T,T) = 1$$

$$= B(t,T)\exp\left(\int_t^T r(s)ds\right) \mathbf{Dex}\left(-\int_t^T \xi dW_0(s)\right)$$

It follows that

$$B(t,T)\, \mathbf{Dex}\left(-\int_t^T \xi dW_0(s)\right) = \exp\left(-\int_t^T r(s)ds\right)$$

$$\Rightarrow \quad B(t,T) = \mathbf{E}_0\left\{\exp\left(-\int_t^T r(s)ds\right) \,\middle|\, \mathrm{F}_t\right\}$$

which is a zero coupon bond

Hedging with futures

Futures contract can directly hedge pre-DI swaps

$$d\,\text{pswap}(t) = d\{B(t,T) - K\,B(t,T_1)\}$$

$$= \text{pswap}(t) r(t) dt + dM_T(t) - K\,dM_{T_1}(t)$$

This result **independent model** ('cos no assumptions about HJM volatility function)

In **HJM framework** can delta hedge swaptions via

$$d\,\text{pswpn}(t)$$

$$= d\{B(t,T)\,\text{BSput}(K\,F_T(t,T_1),\ 1,\ \zeta^2)\}$$

$$= \text{pswpn}(t)\ r(t)dt$$

$$+ \text{BSput}(K\,F_T(t,T_1),\ 1,\ \zeta^2)\ dM_T(t)$$

$$+ \Delta\text{BSput}(K\,F_T(t,T_1),\ 1,\ \zeta^2)$$

$$\times [dM_{T_1}(t) - F_T(t,T_1)\ dM_T(t)]$$

$$\zeta^2 = \int_t^T \left| \int_T^{T_1} \sigma(s,u) du \right|^2 ds$$

In **HJM framework** can delta hedge DI Index options via

$$d\,\mathrm{DIcall}(t,T)$$
$$= d\{\mathrm{BSput}(K\,B(t,T),\ \beta(t),\ \zeta^2)\}$$
$$= \mathrm{DIcall}(t,T)\ r(t)dt$$
$$+ \Delta\mathrm{BSput}(K\,B(t,T),\ \beta(t),\ \zeta^2)\ K\,dM_T(t)$$
$$\zeta^2 = \int_t^T \left| \int_s^T \sigma(s,u)du \right|^2 ds$$

In **BGM framework** can delta hedge swaptions via

$$d\,\mathrm{pswpn}(t)$$
$$= d\{= B(t,T_1)\,\mathrm{BScall}(A(t,T,T_1),K-1,\zeta^2)\}$$
$$= \mathrm{pswpn}(t)\ r(t)dt$$
$$+ \mathrm{BScall}(A(t,T,T_1),K-1,\zeta^2)\ dM_{T_1}(t)$$
$$+ \Delta\mathrm{BScall}(A(t,T,T_1),K-1,\zeta^2)$$
$$\times\,[dM_T(t) - (1 + A(t,T,T_1))dM_{T_1}(t)]$$
$$\zeta^2 = \int_t^T |\gamma(s,T,T_1)|^2 ds$$

DI Futures Options

These are essentially options on zero coupon bonds $B(t, T_1)$ with payoffs like

$$\text{payer}(T) = \left[B(T, T_1) - \frac{1}{[1+\kappa]^{\frac{T_1 - T}{252}}} \right]^+$$

$$= \left[B(T, T_1) - \frac{1}{K} \right]^+$$

where $K = [1 + \kappa]^{\frac{T_1 - T}{252}}$

Hence **market formula** for value payer

$$\text{payer}(t)$$

$$= B(t, T_1)\mathbf{E}_{T_1}\left\{ \frac{1}{B(T, T_1)} \left[B(T, T_1) - \frac{1}{K} \right]^+ \middle| \mathsf{F}_t \right\}$$

$$= \frac{1}{K} B(t, T_1)\mathbf{E}_{T_1}\{[(K-1) - A(T)]^+ | \mathsf{F}_t\}$$

$$= \frac{1}{K} B(t, T_1)\,\text{BScall}(A(t, T, T_1), K - 1, \zeta^2)$$

Formulae almost same those for swaptions so delta hedges into futures contracts similar those for swaptions

Can also clearly use futures options to vega hedge swaptions

Expectation formula

Let X and Y be jointly normally distributed with zero mean, and set $\zeta^2 = \mathrm{var}(X - Y)$. Then

$$C = \mathbf{E}[K\mathbf{e}(X) - L\mathbf{e}(Y)]^+ = K\mathbf{N}(h) - L\mathbf{N}(h - \zeta), \quad \text{w}$$

$$h = \frac{\ln(\frac{K}{L}) + \frac{1}{2}\zeta^2}{\zeta}, \qquad \mathbf{e}(X) = \exp\Big(X - \frac{1}{2}\,\mathrm{var}\,X$$

Greeks are given by

Delta Δ

$$\partial_K C = \mathbf{N}(h), \quad \partial_L C = -\mathbf{N}(h - \zeta)$$

Vega Λ

$$\partial_\zeta C = K\mathbf{N}'(h) = L\mathbf{N}'(h - \zeta)$$

Gamma Γ

$$\partial_K^2 C = \frac{1}{K\zeta}\mathbf{N}'(h), \quad \partial_L^2 C = \frac{1}{L\zeta}\mathbf{N}'(h - \zeta)$$

$$\partial_K \partial_L C = -\frac{1}{L\zeta}\mathbf{N}'(h) = -\frac{1}{K\zeta}\mathbf{N}'(h - \zeta)$$

Black-Scholes formula

From expectation formula

$$
\begin{aligned}
\mathrm{BScall}(S,\kappa,\zeta^2) &= \mathbf{E}[S\mathbf{e}(X) - \kappa]^+ \\
&= S\mathbf{N}(h) - \kappa\mathbf{N}(h-\zeta) \\
\mathrm{BSput}(S,\kappa,\zeta^2) &= \mathbf{E}[\kappa - S\mathbf{e}(X)]^+ \\
&= \kappa\mathbf{N}(-(h-\zeta)) - S\mathbf{N}(-h) \\
h &= \frac{\ln(\frac{S}{\kappa}) + \frac{1}{2}\zeta^2}{\zeta} \\
\zeta^2 &= \mathrm{var}X
\end{aligned}
$$

with deltas

$$
\Delta\mathrm{BScall}(S,\kappa,\zeta^2) = \partial_S \mathrm{BScall}(S,\kappa,\zeta^2) = \mathbf{N}(h)
$$

$$
\Delta\mathrm{BSput}(S,\kappa,\zeta^2) = \partial_S \mathrm{BSput}(S,\kappa,\zeta^2) = -\mathbf{N}(-h)
$$

Arb-free SDE

Let $f(t,X_t)$ be an arbitrage free instrument in which the driver X_t is a diffusion with SDE

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_0(t)$$

under the spot measure P_0.

Because $\beta(t) = \exp\left(\int_0^t r(s)ds\right)$ is a finite variation process, $\frac{f(t,X_t)}{\beta(t)}$ is a P_0 martingale, and stochastic and drift parts on both sides of an equation must match

$$d\left\{\frac{f(t,X_t)}{\beta(t)}\right\} = (\text{expression})dW_0(t)$$

$$= \frac{1}{\beta(t)}\frac{\partial}{\partial x}f(t,X_t)\,\sigma(t,X_t)dW_0(t)$$

Hence the following SDE for $f(t,X_t)$

$$df(t,X_t) = f(t,X_t)r(t)dt + \frac{\partial}{\partial x}f(t,X_t)\,\sigma(t,X_t)dW_0(t)$$