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New Approach to Price Put and Call Options with Discrete Dividends

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nab Clydesdale Bank Yorkshire Bank bnz Great Western Bank MLC

Alex Buryak

Alex has a **PhD** in Physics/Engineering (ANU,1998) and an Executive **MBA** (AGSM, 2005).

Alex was awarded **Gold Bragg Medal 1999** – for the best PhD Thesis in Physics in Australia.

His professional and entrepreneurial endeavours include maintaining communication links in Goby desert, being Maths lecturer in ADFA, advancing state-of-the-art physics and technology research with universities and start-ups, working as a management consultant in Asia and New Zealand, and since 2008 doing a hard core quant work with NAB in Sydney MRQS group specialising in equity and commodity models.

Alex is married with 2 children, 1 cat and ever changing number of mice. His other interests include running, chess and history.

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Equation to Solve

- We concentrate on analysing the stock process S_t which jumps down by the amounts of dividend d_i at the respective times t_i, and follows a geometric Brownian motion with flat volatility σ at other times:

$$dS_t = \left(rS_t - \sum_{0<t_i\leq T} d_i \delta(t - t_i) \right) dt + \sigma\, S_t dW_t. \qquad (*)$$

Here r is the risk-free interest rate, δ is the Dirac delta function and dW_t is a Wiener process

- We are interested in pricing of the corresponding European Put and Call options

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History of the Question

Issue was around for about a decade…

- Literature overview (2002 – 2005)
  - **V. Frishling, "A discrete question",** Risk, Jan. 2002, pp. 115–116 : the issue is identified; two "traditional" approaches are presented; both disagree significantly with numerical results.
  - **M. Bos and S. Vandermark, "Finessing fixed dividends",** Risk, Sept. 2002, pp. 157–158: an idea to split dividends into two appropriately tailored parts (one to be subtracted from S and one to be added to K in BS formula).
  - **R. Bos, A. Gairat, and A. Shepeleva, "Dealing with discrete dividends",** Risk, Jan. 2003, pp. 109–112: volatility adjustment idea; two ways of volatility adjustment are suggested.
  - **R. Beneder and T. Vorst, (2001) "Options on dividend paying stocks",** in: J. Yong (Ed.) Recent Developments in Mathematical finance (Shanghai, 2001) (River Edge, NJ: World Scientific Publishing): a simpler version of volatility adjustment idea (plus only one way to adjust volatility is suggested).

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Case of Distributed Dividends: BS formula

If no discrete dividends are present, then a famous exact solution is known

- Nobel prise winning Black-Scholes formula

$$C = S_0 \exp(-qT) N(d_1) - K \exp(-rT) N(d_2),$$

$$P = K \exp(-rT) N(-d_2) - S_0 \exp(-qT) N(-d_1),$$

where S_0 is a current stock price (Spot), K is Strike value, T is time to option Expiry, r is a risk-free rate, q is dividend yield, $N(x)$ is cumulative normal distribution function and d_1 and d_2 are given by:

$$d_1 = \frac{\ln(S_0 / K) + (r - q + \sigma^2 / 2)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\ln(S_0 / K) + (r - q - \sigma^2 / 2)T}{\sigma\sqrt{T}}.$$

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Option Pricing for Discrete Dividend Case

Basic pricing strategies include adjustment of spot (S) or strike (K);

- Spot approximation (same answer as distributed dividend BS formula):

$$C = \tilde{S}_0 N(d_1) - K\exp(-rT)N(d_2),$$

$$P = K\exp(-rT)N(-d_2) - \tilde{S}_0 N(-d_1),$$

where $\tilde{S}_0 = S_0 - \sum_{0<t_i\le T} d_i \exp(-r\,t_i) = S_0 \exp(-qT).$

- Strike approximation:

$$C = S_0 N(d_1) - \tilde{K}\exp(-rT)N(d_2),$$

$$P = \tilde{K}\exp(-rT)N(-d_2) - S_0 N(-d_1),$$

where $\tilde{K} = K + \sum_{0<t_i\le T} d_i \exp(r(T-t_i)).$

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More Advanced Strategies: Hybrid Approach

One of more advanced approaches uses dividend splitting technique

- Dividend splitting approach [Bos & Vandermark (2002)]:

$$C = \overline{S}_0 N(d_1) - \overline{K}\exp(-rT)N(d_2),$$

$$P = \overline{K}\exp(-rT)N(-d_2) - \overline{S}_0 N(-d_1),$$

where $\overline{S}_0 = S_0 - D_S,\ \overline{K} = K + D_K \exp(rT),$ where, in turn,

$$D_S = \sum_{0<t_i\le T} \frac{T - t_i}{T} d_i \exp(-r\,t_i), \quad D_K = \sum_{0<t_i\le T} \frac{t_i}{T} d_i \exp(-r\,t_i).$$

We refer to this method as Hybrid approach below

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Local versus Implied Volatility

Volatility in BS formulas is related (but not identical!) to volatility in Equation (*)

- Explicit expression linking implied (BS) volatilities with local (MC) volatilities can be derived for distributed dividend case (generalised Dupire formula):

$$\sigma_L^2(S,t) = \left. \frac{\sigma^2 + 2T\sigma \frac{\partial \sigma}{\partial T} + 2(r_f - q_f)KT\sigma \frac{\partial \sigma}{\partial K}}{\left(1 + Kd_1\sqrt{T}\frac{\partial \sigma}{\partial K}\right)^2 + K^2 T\sigma\left(\frac{\partial^2 \sigma}{\partial K^2} - d_1\left(\frac{\partial \sigma}{\partial K}\right)^2\sqrt{T}\right)} \right|_{K=S,\ T=t} , \qquad (**)$$

where $d_1 = \frac{\ln(S/K) + (r_f - q_f + \sigma^2/2)T}{\sigma\sqrt{T}}$ and $\sigma \equiv \sigma_{impl}(K,T)$.

No general formula is known for discrete dividend case, but it can be shown that flat local volatility of process with jumps should be replaced by non-flat local volatility if the "equivalent" process without jumps is considered.

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More Advanced Strategies: Vol Adjustment

More advanced approaches use volatility adjustment idea [Beneder & Vorst, (2001)] [Bos *et. al*., (2003)]

- Volatility adjustment (Spot version)

$$C = \tilde{S}_0 N(d_1) - K\exp(-rT)N(d_2),$$

$$P = K\exp(-rT)N(-d_2) - \tilde{S}_0 N(-d_1),$$

where $\tilde{S}_0 = S_0 - \sum_{0<t_i\le T} d_i \exp(-r\,t_i),\quad \bar{\sigma}_S^2 = \sigma^2\left\langle\left(\frac{S}{S - D_j^{(S)}}\right)^2\right\rangle \equiv \sigma^2(1+\varepsilon_S)^2$

- Volatility adjustment (Strike version)

$$C = S_0 N(d_1) - \tilde{K}\exp(-rT)N(d_2),$$

$$P = \tilde{K}\exp(-rT)N(-d_2) - S_0 N(-d_1),$$

where $\tilde{K} = K + \sum_{0<t_i\le T} d_i \exp(r(T-t_i)),\quad \bar{\sigma}_K^2 = \sigma^2\left\langle\left(\frac{S}{S + D_j^{(K)}}\right)^2\right\rangle \equiv \sigma^2(1-\varepsilon_K)^2$

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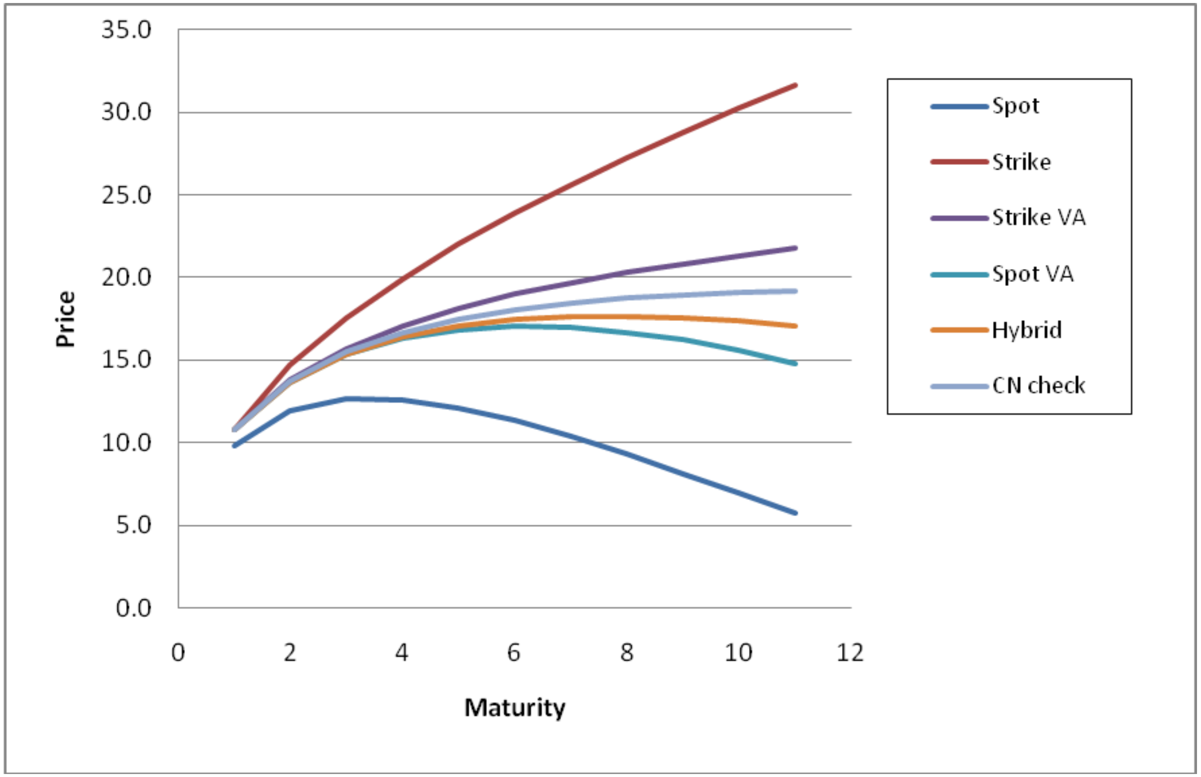
Comparison of Different Methods: Calls

All methods produce results which may significantly differ from numerics



Comparison of results given by different analytical approaches for a multiple dividend family of Calls: $T = 1,\ 2,\ 3,\ \ldots\ ,\ 11;\ S_0 = K = 100;\ \sigma = 30\%;\ r = 6\%;$ $t_i = 0.5,\ 1.5,\ \ldots\ ,\ 10.5;\ D_i = 9.0,\ 9.0,\ \ldots\ ,\ 9.0.$

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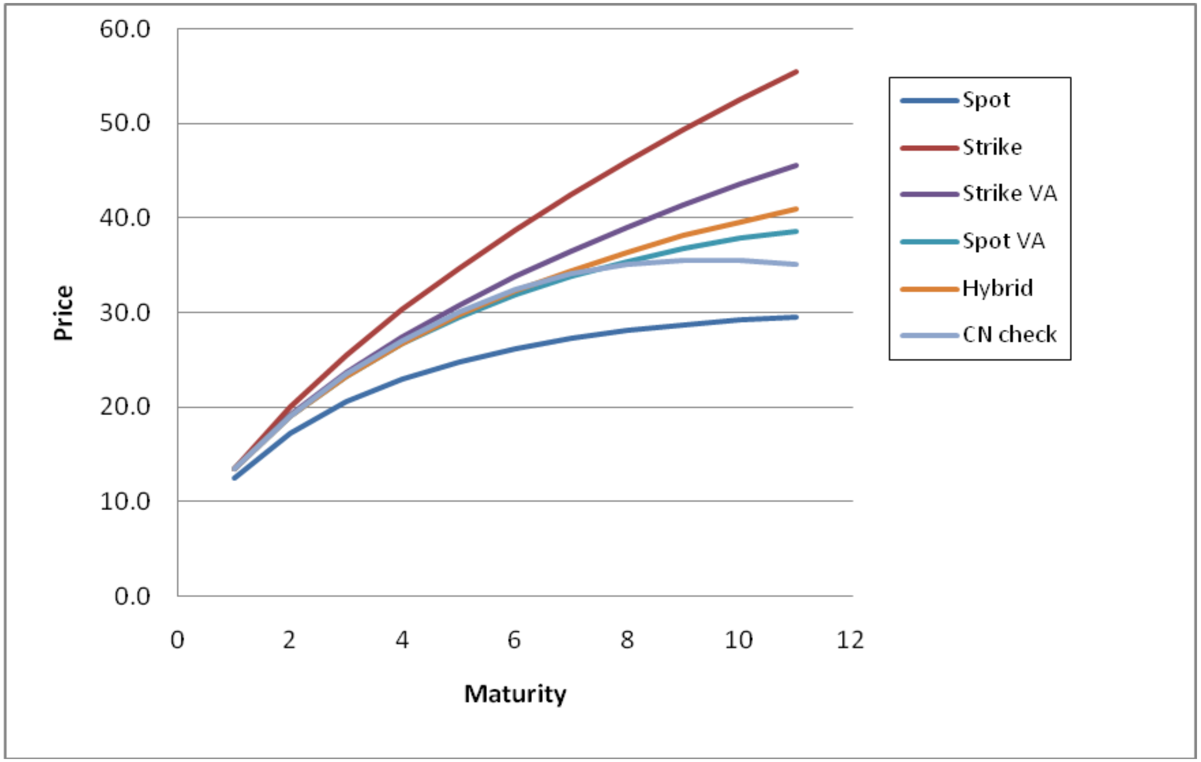
Comparison of Different Methods: Puts

All methods produce results which may significantly differ from numerics



Comparison of results given by different analytical approaches for a multiple dividend family of : $T=1,\ 2,\ 3,\ \ldots\ ,\ 11;\ S_0=K=100;\ \sigma=30\%;\ r=6\%;$ $t_i\ =0.5,\ 1.5,\ \ldots\ ,\ 10.5;\ D_i\ =9.0,\ 9.0,\ \ldots\ ,\ 9.0.$

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Main Observations

- Significant differences with Crank-Nicolson (CN) results
  - Moderate to strong disagreement with any of analytic approximations for Calls
  - Qualitatively different behaviour for CN results vs. analytic approximations for Puts
  - Violation of the conventional Put-Call parity relation for European options:

$$P + S\exp(-q\,T) = P + S - D = C + K\exp(-r\,T),$$

where $$D = \sum_{0<t_i\le T} d_i \exp(-r\,t_i).$$

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Questions to Answer

- Main question
  - Can we trust our CN scheme?
  - Is the observed parity violation phenomenon real?
- Questions for BS-like formulas / approximations
  - Are there any good analytic formulas / approximations?
  - If not, can we develop one?
  - If parity violation phenomenon is real, then can it also be described by BS-like formulas?

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CN schemes - Introduction

CN schemes are still in the centre of attention of a broad quant community

- Equation to solve (from $t = T$ to $t = 0$.)

$$\frac{\partial V}{\partial t} - rV + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0$$

- Initial conditions
  - For Puts $V(T) = \max(K - S_i, 0),$
  - For Calls $V(T) = \max(S_i - K, 0).$
- American versus European schemes
  - European CN solvers utilise LU schemes (direct matrix inversion).
  - American CN solvers typically utilise projected successive over-relaxation (PSOR) method

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Different Versions of BCs for CN schemes

Surprisingly it is harder to formulate BCs for European options

- Boundary conditions for American Puts:

$$V_N(0 \le t \le T) = 0, \quad V_1(0 \le t \le T) = K - S_1.$$

- Upper boundary conditions for European Puts: $V_N(0 \le t \le T) = 0,$

- Lower boundary conditions for European Puts (3 potential versions)

  - "Spot": $V_1(0 \le t \le T) = K\exp(-r(t,T)(T-t)) - \max(S_1 - \tilde{D}(t), 0),$
  - "Strike": $V_1(0 \le t \le T) = [K\exp(-r(t,T)(T-t)) + \tilde{D}(t)] - S_1,$
  - "Hybrid": $V_1(0 \le t \le T) = [K\exp(-r(t,T)(T-t)) + \tilde{D}_K(t)] - \max(S_1 - \tilde{D}_S(t), 0),$

where, e.g. $\tilde{D}(t) = \sum_{t \le t_i \le T} d_i \exp(-r(t,t_i)\ (t_i - t)).$

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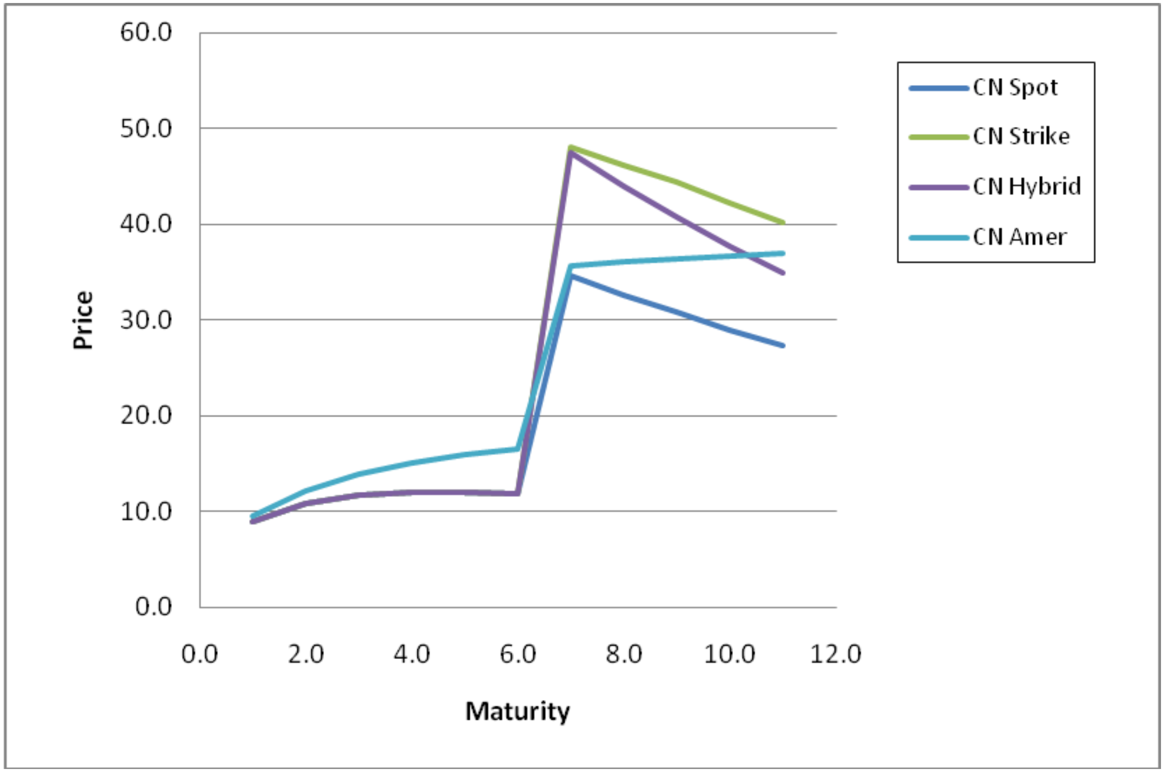
Which CN Euro Option BCs are Correct?

Comparison of three versions of Euro Put families with corresponding American Put family clearly defines the only consistent choice – “Spot” BCs



Only “spot” BCs always keep Euro option values lower than the American ones

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Other CN issues

Dividend-induced CN grid shifts have to be handled with extra care

- As we work backward on the grid through a policy-dependant dividend payment at dividend times the asset price rises by the amount of the dividend so that we have introduce the shift ,

$$S(t_i^-) = S(t_i^+) + \tilde{d}_i,$$

  which, due to continuity of derivative price requirement, in turn, leads to the following equations

  - For European scheme $V[S(t_i^-), t_i^-] = V[S(t_i^+) - \tilde{d}_i, t_i^+],$
  - For American scheme $V[S(t_i^-), t_i^-] = \max(V[S(t_i^+) - \tilde{d}_i, t_i^+], K - S(t_i^+)).$

- All these equations depend on a choice of dividend policy!

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Different Dividend Policies

What if at some point of time the process share price falls below the corresponding forecasted dividend value? Different strategies can be suggested:

- "Liquidator" policy – to payout "everything available":

$$D(t) = \min[\, D^{(forcasted)}(t), S(t)].$$

- "Survivor" policy – to pay nothing:

if $D^{(forcasted)}(t) > S(t)$ than $D(t) = 0.$

- "In-between" policies:

$$0 < D(t) < \min[\, D^{(forcasted)}(t), S(t)].$$

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Monte Carlo Comparison

For European options the standard Monte Carlo algorithm is substantially simpler and may even be more efficient than some CN implementations.

- Standard forward propagation formulas to ex-dividend dates:

$$S_n(t+\Delta t) = S_n(t)\exp[(r-\sigma^2/2)\Delta t + \sigma(\Delta t)^{1/2} z_n],$$

- are followed by dividend subtractions (according to a chosen policy):

$$S(t_i^+) = S(t_i^-) - \tilde{D}_i .$$

- Final results are obtained by averaging of the discounted trajectory payouts:

$$V_C = [\sum_{1\le n\le N} \exp(-rT)\max(S_T - K, 0)]/N, \quad V_P = [\sum_{1\le n\le N} \exp(-rT)\max(K - S_T, 0)]/N$$

- Good agreement with CN results (for “Spot” boundary condition choice)

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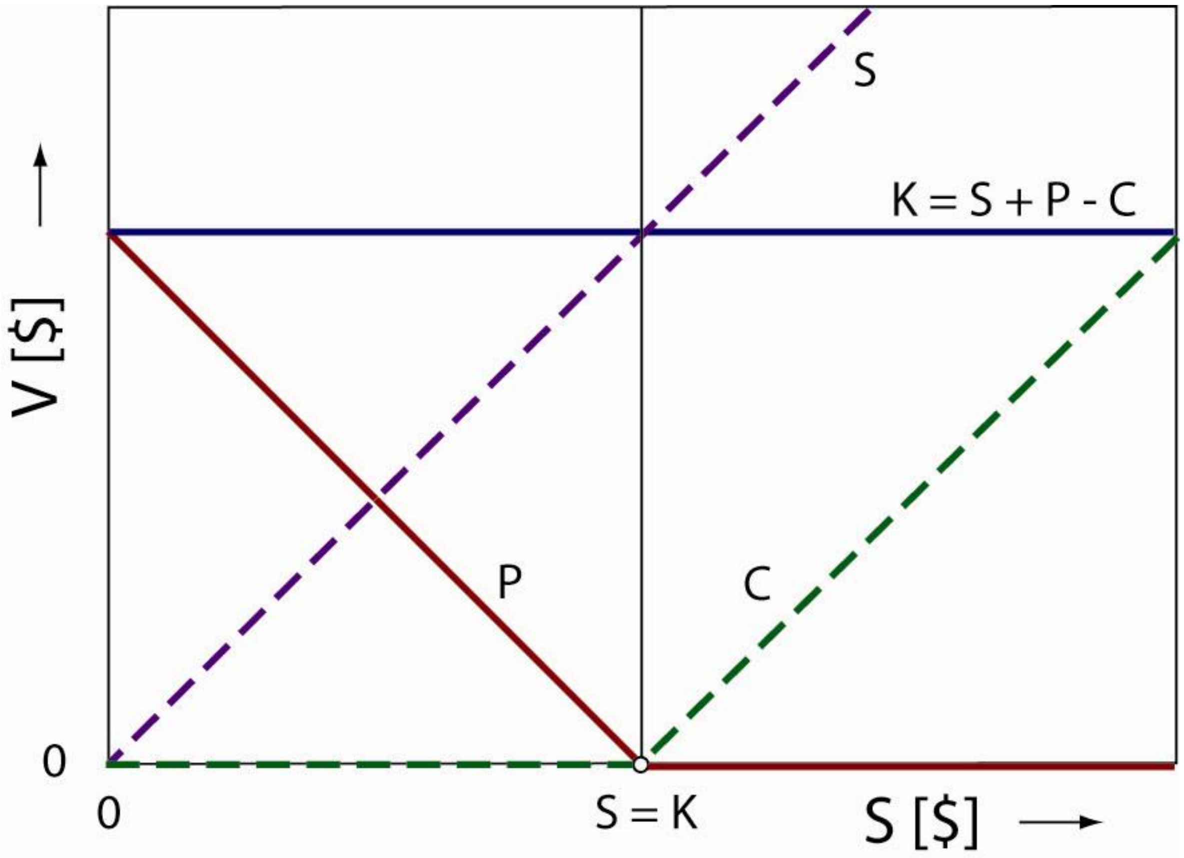
Conventional Put-Call Parity Expression

There is no parity violation in case of zero dividends or proportional dividends:



The relation $P + S\exp(-q\,T) - C = K\exp(-r\,T)$ holds.

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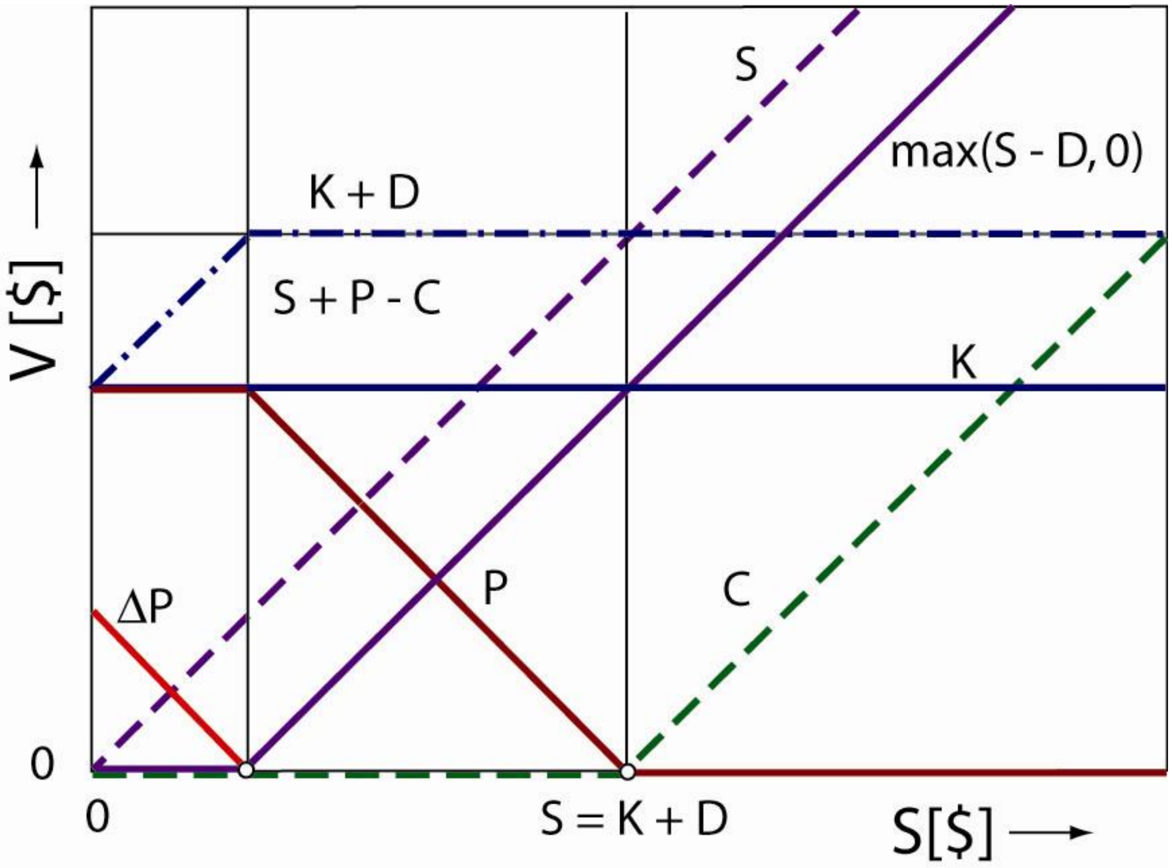
Parity Violation Phenomenon (Liquidator case)

In case of a single discrete dividend forecasted parity violation is almost apparent



Here $D_{actual} = \min(S, D_{forecasted})$, which results in $P + S - D - C < K\exp(-rT)$.

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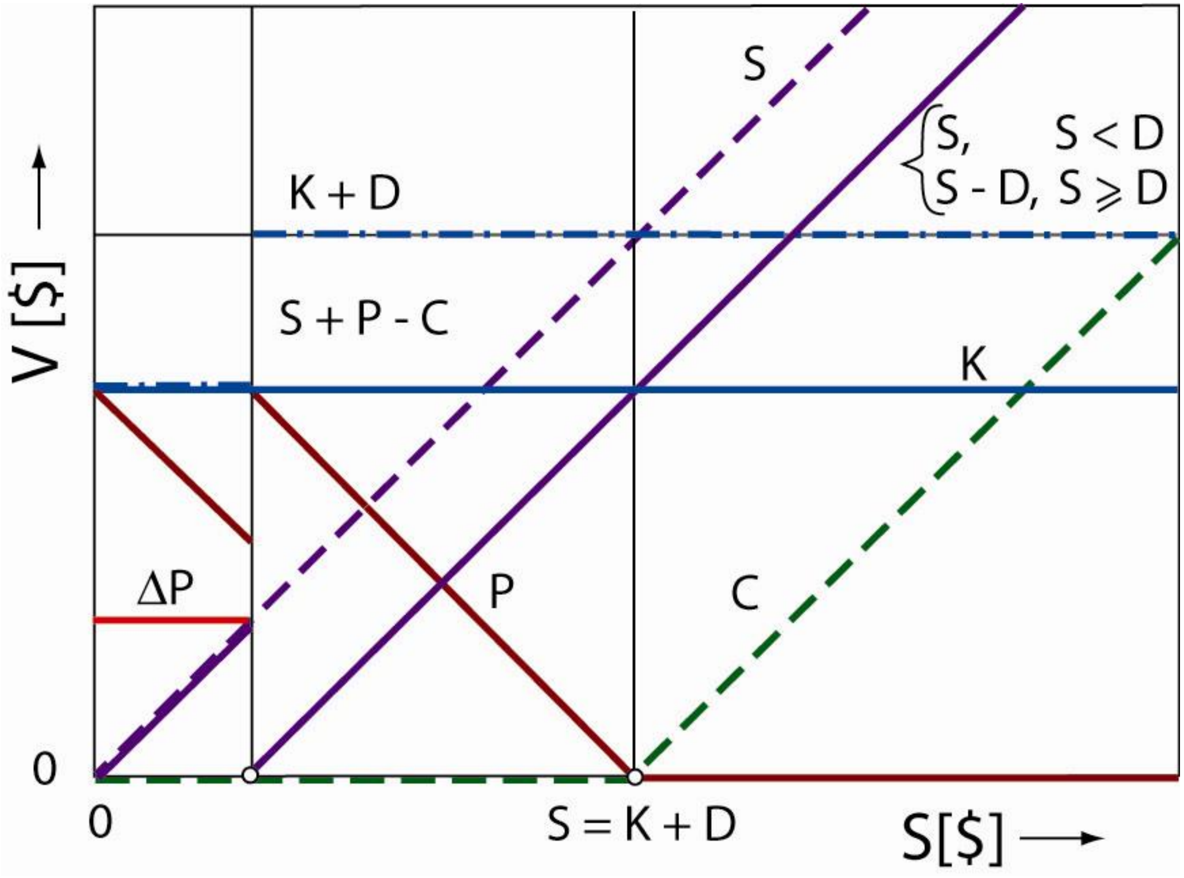
Parity Violation Phenomenon (Survivor case)

There exists more than one dividend payout strategy (“Liquidator” vs. “Survivor”)



Here $D_{actual} = 0$ if $D_{forecasted}(t) > S(t)$.

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Was Parity Violation Overlooked by Others?

We could identified a SINGLE paper, where this phenomenon was reported

- **E. G. Haug, J. Haug, A. Lewis, "Back to Basics: a New Approach to the Discrete Dividend Problem",** Wilmott Magazine, Sept. 2003, 37–47:
  - essentially provides a semi-numerical method of pricing European and American options; method is hard to implement in case of many forecasted dividend payouts
- Other related recent works (2006 - 2010)
  - **M. H. Vellekoop and J. W. Nieuwenhuis, "Efficient pricing of derivatives on assets with discrete dividends",** Applied Math. Finance, Vol. 13, No. 3, Sept. 2006, pp. 265–284: a good overview of existing pricing methods/approximations; does NOT mention parity violation.
  - **C. Veiga, U. Wystup, "Closed formula for options with discrete dividends and its derivatives",** Applied Math. Finance, Vol. 16, Issue 6, 2009, pp. 517-531: provides an exact Tailor series-like formula, which works well for multi-dividend case; again NO mention about parity violation.
- Reasons for missing Parity Violation phenomenon
  - Most researches are more focused on American options.
  - 100% of results are presented for Calls ONLY, whereas parity violation manifests itself ONLY in Put pricing.

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Improved Pricing of Euro Calls: Core Idea

It may be possible to construct a simple, but high accuracy BS-like approximation

- Existing approaches
  - Spot adjusted approach (Spot) + extra volatility adjusted approach (VA Spot)
  - Strike adjusted approach (Strike) + extra volatility adjusted approach (VA Strike)
  - Dividend-split approach (Hybrid)
- Standard perturbation theory framework
  - Standard perturbation approach involves having a zero-order approximation and tuning parameter(s) which can be adjusted once or many times (i.e. in a series-like fashion).
- Action plan
  - Take dividend-split approach as a zero-order approximation
  - Apply an extra volatility adjustment in a fashion prescribed by VA Spot and VA Strike approaches
  - Enjoy much improved agreement

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Improved Pricing of Euro Calls: Analytics

Newly suggested method use both dividend splitting and volatility adjustment

- VA Hybrid approach:

$$C = \bar{S}_0 N(d_1) - \bar{K}\exp(-rT)N(d_2),$$

$$P = \bar{K}\exp(-rT)N(-d_2) - \bar{S}_0 N(-d_1),$$

where $\bar{S}_0 = S_0 - D_S,\ \bar{K} = K + D_K \exp(rT),$ where, in turn,

$$D_S = \sum_{0<t_i\le T} \frac{T-t_i}{T} d_i \exp(-r\,t_i),\quad D_K = \sum_{0<t_i\le T} \frac{t_i}{T} d_i \exp(-r\,t_i).$$

and we also adjust volatility as $\bar{\sigma}_H = \sigma(1+\varepsilon_S^{(h)})(1-\varepsilon_K^{(h)}) \equiv \sigma(1+\varepsilon_H),$

$$(1-\varepsilon_K^{(h)}) \equiv \sqrt{\frac{1}{T}\left[t_1 + \sum_{1<j<N}\left(\frac{S}{S+D_j^{(K)}}\right)^2 (t_j - t_{j-1}) + \left(\frac{S}{S+D_N^{(K)}}\right)^2 (T-t_N)\right]},$$

$$(1+\varepsilon_S^{(h)}) \equiv \sqrt{\frac{1}{T}\left[\left(\frac{S}{S-D_1^{(S)}}\right)^2 t_1 + \sum_{1<j<N}\left(\frac{S}{S-D_j^{(S)}}\right)^2 (t_j - t_{j-1}) + (T-t_N)\right]}.$$

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Improved Pricing of Euro Calls: Many Div Case

Hybrid VA approach provides close to perfect agreement with CN results

T	CN	Spot VA	Rel diff	Strike VA	Rel diff	Hybrid	Rel diff	TE	Rel diff	Hybrid VA	Rel diff
yrs	[$]	[$]	[%]	[$]	[%]	[$]	[%]	[$]	[%]	[$]	[%]
1	10.19	10.2	0	10.23	0.3	10.18	-0.1	10.19	0	10.2	0.1
2	13.22	13.15	-0.5	13.33	0.8	13.16	-0.4	13.21	-0.1	13.22	0.1
3	15.04	14.84	-1.4	15.29	1.7	14.91	-0.8	15.02	-0.2	15.05	0
4	16.24	15.82	-2.6	16.67	2.8	16.01	-1.4	16.21	-0.2	16.24	0
5	17.07	16.33	-4.3	17.77	4.1	16.7	-2.2	17.04	-0.2	17.05	-0.2
6	17.66	16.51	-6.5	18.64	5.5	17.11	-3.1	17.62	-0.2	17.6	-0.4
7	18.08	16.41	-9.2	19.37	7.1	17.31	-4.3	18.03	-0.3	17.96	-0.7
8	18.37	16.06	-12.5	19.99	8.8	17.34	-5.6	-	-	18.17	-1.1
9	18.57	15.56	-16.2	20.53	10.5	17.25	-7.1	-	-	18.27	-1.7
10	18.72	14.87	-20.6	21.02	12.3	17.06	-8.8	-	-	18.27	-2.4
11	18.81	14.01	-25.5	21.47	14.1	16.79	-10.7	-	-	18.21	-3.2

Comparison of results given by different analytical approaches for a multiple dividend family of Calls: $T = 1,\ 2,\ 3,\ \ldots\ ,\ 11;\ S_0 = K = 100;\ \sigma = 30\%;\ r = 6\%;$ $t_i = 0.5,\ 1.5,\ \ldots\ ,\ 10.5;\ D_i = 9.0,\ 9.0,\ \ldots\ ,\ 9.0.$

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Improved Pricing of Euro Calls: One Div Case

VA Hybrid approach provides close to perfect agreement with CN results

T	CN	Spot VA	Rel diff	Strike VA	Rel diff	Hybrid	Rel diff	TE	Rel diff	Hybrid VA	Rel diff
yrs	[$]	[$]	[%]	[$]	[%]	[$]	[%]	[$]	[%]	[$]	[%]
1	2.18	3.14	43.9	2.18	0.2	2.18	0.1	5.86	168.6	2.19	0.2
2	4.42	5.07	14.6	4.65	5.3	3.85	-12.9	4.7	6.3	4.46	1
3	6.76	7.12	5.9	7.4	10.1	5.9	-12.2	5.69	-15.3	6.81	1.2
4	9.01	9.22	2.3	10.23	13.6	8.08	-10.3	7.39	-17.9	9.14	1.5
5	11.23	11.3	0.6	13.05	16.2	10.28	-8.5	9.35	-16.7	11.41	1.6
6	13.38	13.34	-0.3	15.81	18.2	12.44	-7	11.4	-14.8	13.6	1.6
7	15.45	15.33	-0.8	18.5	19.8	14.55	-5.8	13.45	-13	15.7	1.6
8	17.43	17.25	-1	21.09	21	16.59	-4.8	15.46	-11.3	17.7	1.5
9	19.32	19.1	-1.2	23.58	22	18.54	-4	17.4	-9.9	19.61	1.5
10	21.12	20.86	-1.2	25.96	22.9	20.4	-3.4	19.28	-8.7	21.42	1.4
11	22.84	22.56	-1.2	28.25	23.7	22.19	-2.9	21.08	-7.7	23.15	1.4

Comparison of results given by different analytical approaches for single dividend family of Calls: $T = 1,\ 2,\ \dots\ ,\ 11; S_0 = K = 100; \sigma = 30\%; r = 6\%; t_1 = \frac{364}{365}, D_1 = 50.$

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TE approach overview

In our opinion, TE approach^ is the only currently available viable analytic alternative to our method.

- Advantages
  - High quality closed form approximation for the majority of considered examples
  - Can programmed to be reasonably fast
- Drawbacks
  - Relative complexity: even in its second order approximation “the number of evaluations of the Black-Scholes pricing formula or any of its derivatives amounts to 2187”
  - Sometimes fails badly (see our single-dividend example for Calls)
  - In its original form misses parity violation phenomenon

^ **C. Veiga, U. Wystup, “Closed formula for options with discrete dividends and its derivatives”,** Applied Math. Finance, Vol. 16, Issue 6, 2009, pp. 517-531

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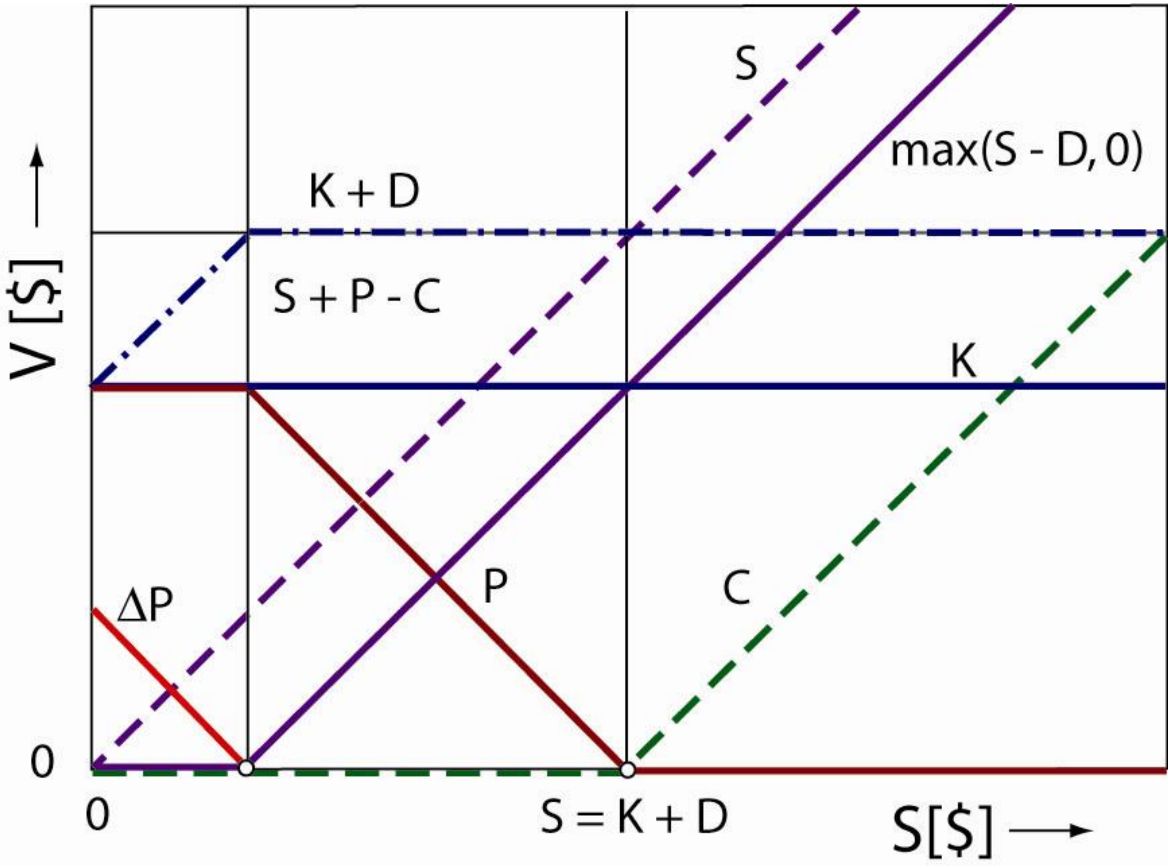
Parity Result for Liquidator Dividend Policy

For a single dividend case exact result for Parity Violation (PV) can be obtained



PV result: $\Delta P = K\exp(-rT)N(-d_2) - S_0 N(-d_1),$ where $K = D,\ T = t_d.$

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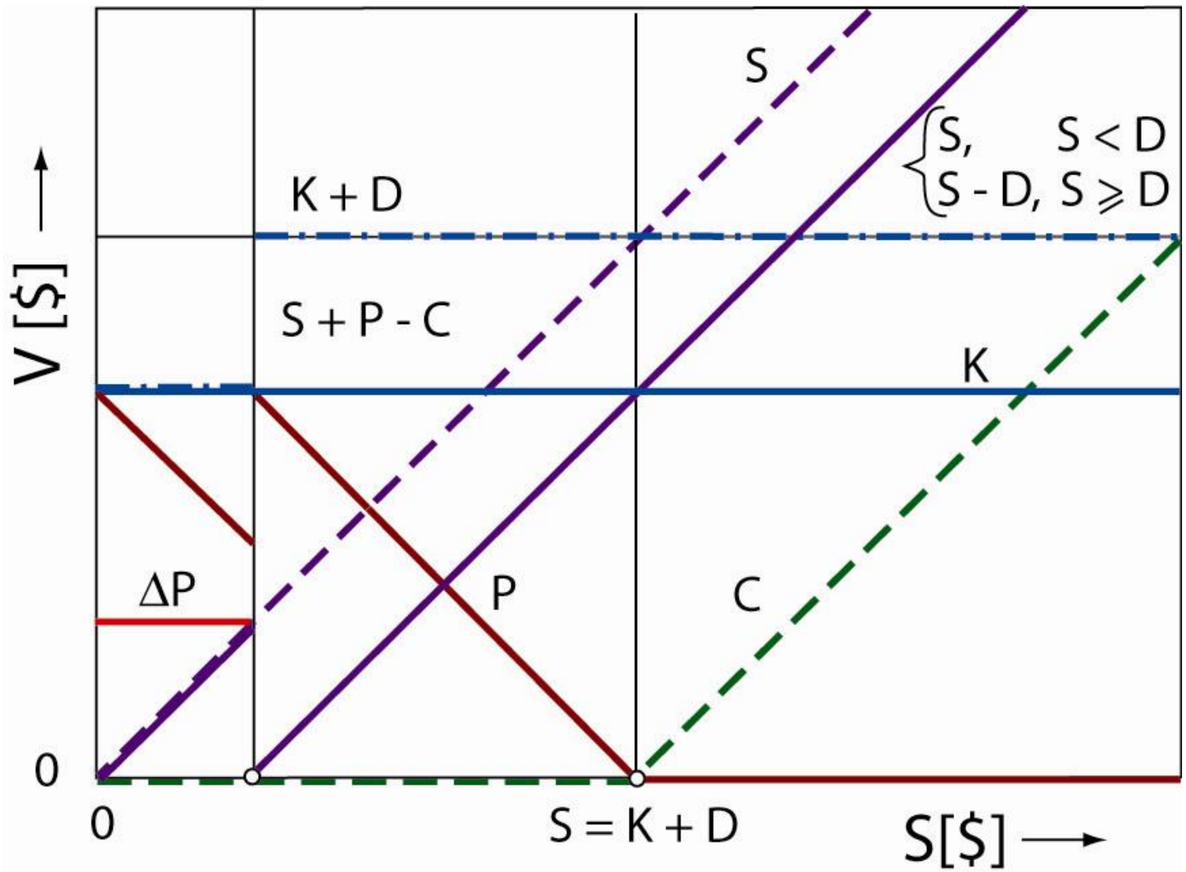
Parity Result for Survivor Dividend Policy

For a single dividend case exact result for Parity Violation (PV) can be obtained



PV result: $\Delta P = K \exp(-rT) N(-d_2)$, where $K = D,\ T = t_d$.

Parity Violation: Multi-Dividend Generalisation

We can utilise newly developed Hybrid VA approach to calculate PV adjustment

- PV correction can be viewed as a Put of term T_D (defined by the timing of the last dividend D_L) with a strike $\overline{D}_L$ given by the adjusted sum of all dividends:

$$\Delta P = \overline{D}_L \exp(-rT_D) N(-d_2) - \overline{S}_0 N(-d_1),$$

where $\overline{S}_0 = S_0 - \widetilde{D}_S$, $\overline{D}_L = D_L + \widetilde{D}_K \exp(rT_D)$, where, in turn,

$$\widetilde{D}_S = \sum_{0<t_i<T_D} \frac{T_D - t_i}{T_D} d_i \exp(-r\, t_i), \quad \widetilde{D}_K = \sum_{0<t_i<T_D} \frac{t_i}{T_D} d_i \exp(-r\, t_i).$$

and we also adjust volatility as $\overline{\sigma}_H = \sigma(1+\varepsilon_S^{(h)})(1-\varepsilon_K^{(h)}) \equiv \sigma(1+\varepsilon_H)$,

$$(1-\varepsilon_K^{(h)}) \equiv \sqrt{\frac{1}{T_D}\left[t_1 + \sum_{1<j<N} \left(\frac{S}{S+\widetilde{D}_j^{(K)}} \right)^2 (t_j - t_{j-1}) + \left(\frac{S}{S+\widetilde{D}_N^{(K)}} \right)^2 (T_D - t_N) \right]},$$

$$(1+\varepsilon_S^{(h)}) \equiv \sqrt{\frac{1}{T_D}\left[\left(\frac{S}{S-\widetilde{D}_1^{(S)}} \right)^2 t_1 + \sum_{1<j<N} \left(\frac{S}{S-\widetilde{D}_j^{(S)}} \right)^2 (t_j - t_{j-1}) + (T_D - t_N) \right]}.$$

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Improved Put Results: Multi-dividend Case

Hybrid VAPA approach provides close to perfect agreement with CN results

T	CN	Hybrid	Rel diff	Hybrid PA	Rel diff	TE	Rel diff	TE PA	Rel diff	Hybrid VA	Rel diff	Hybrid VAPA	Rel diff
yrs	[$]	[$]	[%]	[$]	[%]	[$]	[%]	[$]	[%]	[$]	[%]	[$]	[%]
1	13.1	13.09	-0.1	13.09	-0.1	13.1	0	13.1	0	13.11	0.1	13.11	0.1
2	18.86	18.81	-0.3	18.81	-0.3	18.86	0	18.86	0	18.87	0.1	18.87	0.1
3	23.25	23.13	-0.5	23.13	-0.5	23.25	0	23.25	0	23.26	0	23.26	0
4	26.87	26.66	-0.9	26.64	-0.9	26.88	0	26.88	0	26.89	0	26.87	0
5	29.92	29.64	-0.9	29.5	-1.3	29.99	0.3	29.96	0.1	29.98	0.2	29.84	-0.3
6	32.33	32.2	-0.4	31.69	-2	32.73	1.2	32.44	0.3	32.69	1.1	32.13	-0.6
7	34.06	34.42	1	33.21	-2.5	35.17	3.2	34.19	0.4	35.07	2.9	33.74	-1
8	35.12	36.36	3.5	34.1	-2.9	-	-	-	-	37.19	5.9	34.7	-1.2
9	35.59	38.07	6.9	34.47	-3.1	-	-	-	-	39.09	9.8	35.12	-1.3
10	35.56	39.58	11.3	34.42	-3.2	-	-	-	-	40.79	14.7	35.09	-1.3
11	35.14	40.9	16.4	34.02	-3.2	-	-	-	-	42.32	20.4	34.71	-1.2

Comparison of results given by different analytical approaches for a multiple dividend family of : $T = 1,\ 2,\ 3,\ \ldots\ ,\ 11;\ S_0 = K = 100;\ \sigma = 30\%;\ r = 6\%;$
$t_i = 0.5,\ 1.5,\ \ldots\ ,\ 10.5;\ D_i = 9.0,\ 9.0,\ \ldots\ ,\ 9.0.$

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Parity Violation Results for Survivor Case

There are important differences between Parity Violation adjustments for Liquidator and Survivor case

- Liquidator case (results presented on the previous slide)
  - Any evolutional trajectory which touches zero value due to dividend payout is staying at zero forever:

$$S_n(t+\Delta t) = S_n(t)\exp[(r-\sigma^2/2)\Delta t + \sigma(\Delta t)^{1/2} z_n].$$

  - Thus no recursive sum is present in Parity Violation adjustment formula (effective Put calculations)
  - Non-recursive formula is in good agreement with CN or MC modelling

- Survivor case (and potentially “in-between” policies)
  - Recursive sum generalisation is needed:

$$\Delta P^{(S)}(t_N) = \sum_{1\le j\le N} (-1)^{N-i} P_{eff}^{(S)}(t_i)$$

  - This straightforward recursion leads to relatively poor agreement with numerics - more finetuning is necessary...

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More Rigorous Calculation Approach

The method presented so far is a one-off improvement, which does not allow calculation of higher order correction terms. More general result can be obtained by extending of the approach of R. Bos *et al*, Risk, Jan. 2003, pp. 109–112

- We start from the effective local volatility result

$$\sigma_L(S,t) = \sigma\left(1 + \left[\tilde{D}_S(t) - \tilde{D}_K(t)\right]/S\right),$$

- Then by using semi-group perturbation theory of Schnaubelt (2000)* one can show that the corresponding implied volatility can be approximated as:

$$\sigma_{impl}(K,T) = \frac{1}{T}\int_0^T E\left[\sigma_L\left(\exp(rt + X^{s,l,k}_{\sigma^2 t}), t\right)\right] dt,$$

where $X^{s,l,k}_{\sigma^2 t}$ is the time $\sigma^2 t$ value of a Brownian bridge process from $s = \ln\left(S_0 - \tilde{D}_S(0)\right)$ to $k = \ln\left[\left(K - \tilde{D}_K(T)\right)\exp(-rT)\right]$ of length $l = \sigma^2 T$.

* **R. Schnaubelt**, "Semigroups for nonautonomous Cauchy problems" in Springer Graduate Texts in Mathematics 194, pp. 477-496 (2000).

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More Rigorous Calculations: Final Expression

Using properties of the Brownian bridge the final expression can be obtained:

$$\sigma(K,T)^2 = \frac{\sigma^2}{T}\int_0^T \left(1 + 2\left(\hat{D}_S(t) - \hat{D}_K(t)\right) e^{-x_t + \frac{\sigma^2}{2} y_t} + \left(\hat{D}_S(t) - \hat{D}_K(t)\right)^2 e^{2(-x_t + \sigma^2 y_t)}\right) dt$$

$$= \sigma^2 + 2\sigma\sqrt{\frac{2\pi}{T}} e^{\frac{a^2}{2} - s}\left[\sum_i \alpha_i d_i e^{-rt_i}\left(\Phi(a) - \Phi\left(a - \frac{\sigma t_i}{\sqrt{T}}\right)\right)\right.$$

$$\left. - \sum_i (1-\alpha_i) d_i e^{-rt_i}\left(\Phi\left(a - \frac{\sigma t_i}{\sqrt{T}}\right) - \Phi\left(a - \sigma\sqrt{T}\right)\right)\right]$$

$$+ \sigma\sqrt{\frac{\pi}{2T}} e^{\frac{b^2}{2} - 2s}\left[\sum_{i,j} \alpha_i d_i \alpha_j d_j e^{-r(t_i + t_j)}\left(\Phi(b) - \Phi\left(b - \frac{2\sigma\min(t_i, t_j)}{\sqrt{T}}\right)\right)\right.$$

$$+ \sum_{i,j} (1-\alpha_i) d_i (1-\alpha_j) d_j e^{-r(t_i + t_j)}\left(\Phi\left(b - \frac{2\sigma\max(t_i, t_j)}{\sqrt{T}}\right) - \Phi\left(b - 2\sigma\sqrt{T}\right)\right)$$

$$\left. - 2\sum_{i>j} \alpha_i d_i (1-\alpha_j) d_j e^{-r(t_i + t_j)}\left(\Phi\left(b - \frac{2\sigma t_j}{\sqrt{T}}\right) - \Phi\left(b - \frac{2\sigma t_i}{\sqrt{T}}\right)\right)\right].$$

see our paper for all derivation details and notations...

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Recent Results

Results for Non-flat Curves and Barrier Options

- Results for vanilla options with non-flat local volatility
  - Instead of flat volatility, generalised Dupire formula for local volatility is used
  - Even in its original form of this talk Hybrid VA and Hybrid VAPA approaches agree reasonably well with MC modelling, although the agreement is worse, than for flat volatility case

- Barrier option results
  - Some very preliminary encouraging results for single barrier options
  - More work has to be done...

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Conclusions

A few interesting results have been obtained

- Results
  - Parity Violation phenomenon is confirmed to be real
  - Simple, but high accuracy BS-like results for Euro Calls and Puts are obtained
  - This includes efficient parity violation approximation
  - Due to simple form of the Hybrid VA / Hybrid VAPA approximations the corresponding results for Greeks can be obtained straightforwardly as well

- Possible Future Work
  - Improvement of approximation efficiency by constructing higher order VA correction perturbation series
  - Improvement of parity violation adjustment for Survivor dividend policy case
  - American option binomial tree algorithm generalisation
  - Improved Hybrid VA / Hybrid VAPA approximation for non-flat volatility surfaces
  - Hybrid VA / Hybrid VAPA approximation for other option types
  - Any other suggestions?

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A day to remember: **12-th of April 1961**

50 years of the First Manned Spaceflight – Yuri Gagarin

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