Default Correlation Modelling

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Where is Default Correlation Relevant?

- Letter of Credit backed debt and credit guarantees
- Counterparty risk
  — especially in credit derivatives or with several counterparties
- Portfolios of defaultable bonds
- CDOs and basket structures
- Credit risk diversification

Causes of Default Correlation

- direct relationship between both parties
  e.g. one firm creditor of another
- using same inputs
- general state of a certain industry/region
- general state of the economy

Default Correlations: Basics

Consider, for a fixed time horizon T

Default probabilities of A and B: $p_A, \quad p_B$

For the case of joint defaults, we need

- joint default probability p_{AB}
- conditional default probabilities $p_{A|B}$ and $p_{B|A}$
- correlation ϱ_{AB} between default events $\mathbf{1}_{\{A\}}$ and $\mathbf{1}_{\{B\}}$.

In addition to p_A, p_B, we need least one of the above to calculate the other two.

These values are linked via the definition of conditional probability:

$$p_{A|B} = \frac{p_{AB}}{p_B}, \qquad p_{B|A} = \frac{p_{AB}}{p_A}$$

and correlation

$$\varrho_{AB} = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1 - p_A)p_B(1 - p_B)}}.$$

The Impact of Correlation

Because default probabilities are typically quite small, correlation can have a much larger impact on credit portfolios than, say, equity portfolios.

Consider the joint default probability:

$$p_{AB} = p_A p_B + \varrho_{AB}\sqrt{p_A(1-p_A)p_B(1-p_B)}$$

Supposing that $p_a = p_B \ll 1$ we have: $p_{AB} \approx p^2 + \varrho p \approx \varrho p$

and similarly for the conditional default probability:

$$p_{A|B} = p_A + \varrho_{AB}\sqrt{\frac{p_A}{p_B}(1-p_A)(1-p_B)} \qquad \Rightarrow \qquad p_{A|B} \approx \varrho$$

Market Variables and Data Sources

- **Actual rating and default correlations**

  Advantages: Objective, direct

  Disadvantages: Sparse data sets, long time ranges, need to aggregate

- **Spread correlations**

  Advantages: Reflect information in markets

  Disadvantages: Data quality problems, liquidity, availability

- **Equity correlations**

  Advantages: liquid, easily available, good quality data

  Disadvantages: link to credit quality less obvious, needs a lot of pre-processing

- ***Trac–X* or *iBoxx* tranches**

  Advantages: closest thing to traded “implied correlation”

  Disadvantages: not yet very liquid, reduction of dimensionality still necessary

Need to Reduce Dimensionality

We need to add structure to the models in order to:

- to reduce dimensionality (dimension easily $>$1000)
- to compensate missing data

Free specification of *all* joint default probabilities is too complex:

For N names have $\mathbf{2^N}$ joint default events!

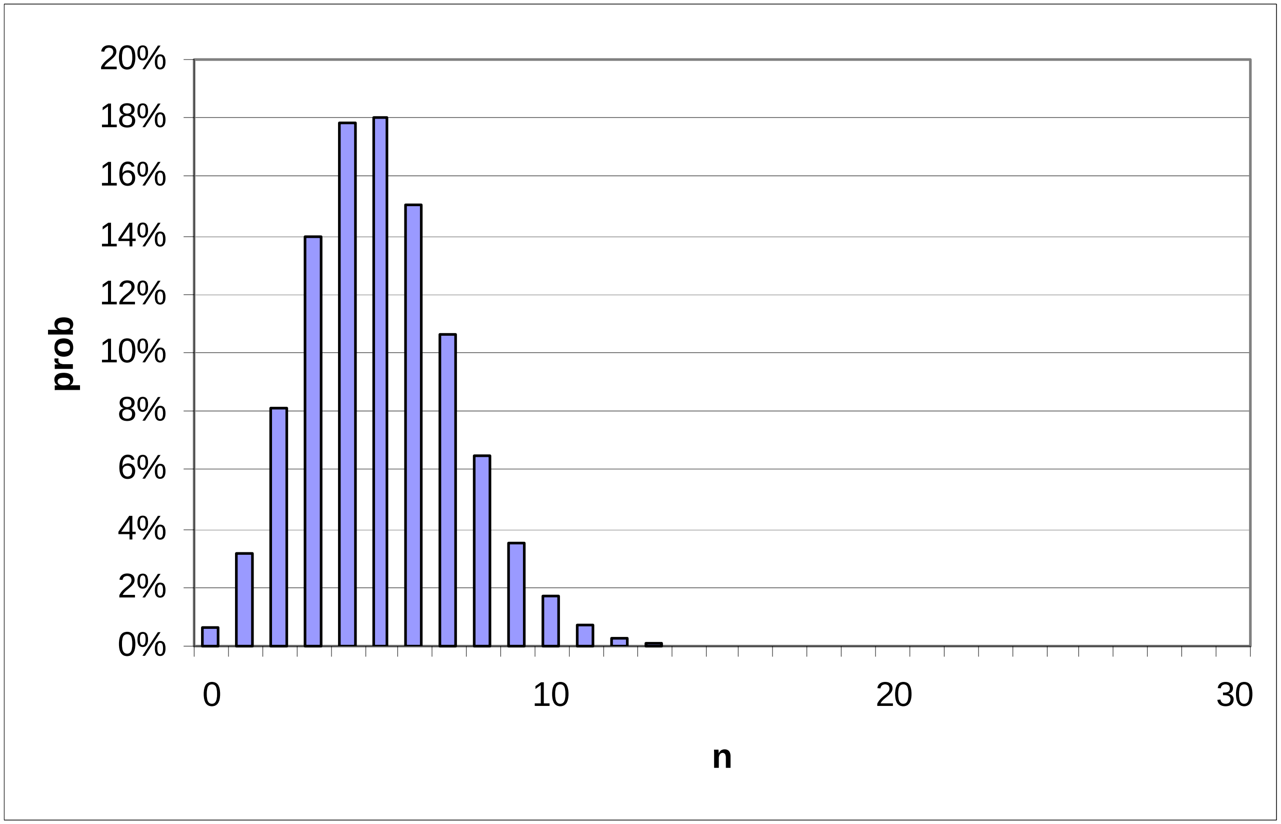
Note that individual default modelling and correlation modelling can be separated.
(Buzzword: Copula Functions)

Simplest Case: Homogeneous Portfolio, Independent Defaults

Consider the losses due to default until a fixed time–horizon T.
For simplicity, interest–rates are set to zero.

- In a portfolio of N exposures to N different obligors,
- with exposures of identical size L,
- assume that defaults happen independently of each other,
- where each obligor defaults with a probability of p before the time-horizon T.

Binomially Distributed Losses



Number of obligors $N = 100$, individual default probability $p = 5\%$

Note that

- The binomial distribution for independent defaults has a very **thin tail**, thus not representing the probability of a large number of defaults realistically.

Default Prob. (%)	1	2	3	4	5	6	7	8	9	10
99.9% VaR Level	5	7	9	11	13	14	16	17	19	20

- If the number of obligors is large, the loss distribution approximates a **normal distribution**.

- At the other extreme, perfect correlation means all or none of the obligors default.

Moody’s Binomial Expansion Method

- For a a Binomial distribution with independent obligors, the tail with fewer (but larger) obligors is “fatter” than the tail with many (but smaller) independent obligors.

- Moody’s are aware, that pure Binomial distribution with independent defaults is unrealistic.

- Make tails of distribution fatter by assuming fewer obligors (the *diversity score*). Adjustments are made for:
  - differences in exposure sizes
  - industry concentration
  - country concentration

Diversity Score

For a given portfolio, this is the number of obligors in an idealised homogeneous comparison portfolio with (cf. Duffie/Gârleanu 2001):

- the same total face value,
- independent defaults,
- comparison bonds that are, in some sense, of the same average default probability as those in the original portfolio, and
- the comparison portfolio has, according to some measure of risk, the same total loss risk as the original portfolio.

Asset–based Approach

(a.k.a. *firm's value* or *structural* model)

- The underlying firm's value $V_n(t)$ is modelled for each obligor n.
- Default of obligor n is triggered when $V_n(T)$ falls below a barrier K_n.
- Can be extended to multiple rating classes by introducing multiple "rating transition barriers" $K_{n,m}$.
  These can be calibrated to a given set of rating transition probabilities (this is very similar to the approach used in CreditMetrics)
- Default correlation is introduced via correlated dynamics of the $V_1, \ldots, V_N$.

Equity price correlations can be used as a proxy for the correlations of the V_n.

Large Homogeneous Portfolio Approximations

Considerable tractability can be gained in the limit for a portfolio of a large number of obligors, homogeneous in

- exposure size
- probability of default
- loss given default
- dependence on underlying factor(s)

E.g. in a one–factor model (Vasicek (1987))

$$V_n = \beta X + \sqrt{1-\beta^2}\epsilon_n, \qquad K_n \equiv K$$

Given $\eta = K - \beta X$, we have for the probability of default

$$p(\eta) = \Phi\left(\frac{\eta}{\sqrt{1-\beta^2}}\right)$$

Invoking the Law of Large Numbers, the loss fraction of the portfolio is given by

$$L \approx (1 - \text{recovery})\Phi\left(\frac{\eta}{\sqrt{1-\beta^2}}\right)$$

Similar results can be derived for other distributions of X, i.e.

- Student–t distribution (O'Kane & Schlögl (2002))
- Archimedean distributions (Schönbucher (2003))

Factor Models for the Asset Dynamics

Assume that I independent factors X_i drive the dynamics of the V_n.

The weight of the i–th factor influencing the n–th obligor is denoted by β_n^i:

$$V_n = \sum_{i=1}^{I} \beta_n^i X_i + \epsilon_n$$

where ϵ_n represents the idiosyncratic risk of firm n, i.e. the ϵ_n are independent of each other and of the X_i.

Obligor n defaults if: $V_n \leq K_n$.

For tractability, the factors and errors are often assumed to be normally distributed.

Difficulties with the Asset–based Approach

- Typically, these models are not calibrated to observed credit spreads. In this respect, they are more suitable for risk–assessment, not pricing.
- Typically, either asset values or returns are assumed to be lognormally or normally distributed (though alternatives have been proposed). This implies problems for the tail of the distribution:
  - thin tails
  - asymptotic tail independence

Irrespective of the distributional assumptions, data problems in determining the correct correlation are only seemingly resolved. **What matters for default correlation is the dependence structure of extreme (low) firm values.**

Default Correlations: Intensity Models

(also called *reduced–form* or *actuarial* models)

Issuer A: Intensity λ_A
Issuer B: Intensity λ_B

λ_A and λ_B are stochastic, correlated
but given the intensities the defaults are still independent.

Correlation introduced indirectly via:

A defaults $\Rightarrow$ λ_A was probably high $\Rightarrow$ λ_B is probably high, too
(due to the correlation between λ_A and λ_B)

$\Rightarrow$ issuer B more likely to default, too.

Joint Default Probability

$$\begin{aligned}
p_{AB} &= \mathbf{E}\left[\ \mathbf{1}_{\{A\}}\mathbf{1}_{\{B\}}\ \right] \\
&= \mathbf{E}\left[\ \mathbf{E}\left[\ \mathbf{1}_{\{A\}}\mathbf{1}_{\{B\}}|\lambda_A\ \right]\ \right] \\
&= \mathbf{E}\left[\ (1 - e^{-\int_0^T \lambda_A ds})\mathbf{1}_{\{B\}}\ \right] \\
&= \mathbf{E}\left[\ (1 - e^{-\int_0^T \lambda_A ds})(1 - e^{-\int_0^T \lambda_B ds})\ \right] \\
&= 1 - (1 - p_A) - (1 - p_B) + \mathbf{E}\left[\ e^{-\int_0^T (\lambda_A + \lambda_B) ds}\ \right] \\
&= p_A + p_B + \mathbf{E}\left[\ e^{-\int_0^T (\lambda_A + \lambda_B) ds}\ \right] - 1.
\end{aligned}$$

Joint Survival Probability

Thus the joint *survival* probability is

$$\mathbf{E}\left[1 - e^{-\int_0^T \lambda_A + \lambda_B ds} \right].$$

Intensities add up for joint survival, i.e.

$$\lambda = \lambda_A + \lambda_B$$

can be viewed as the intensity of the process triggering a default time.

Most extreme correlation if both intensities are perfectly correlated: $\lambda_A = \lambda_B = \lambda$. Then:

$$p_{AB} = 2p + \mathbf{E}\left[\, e^{-2\int_0^T \lambda ds} \,\right] - 1.$$

Then the correlation ϱ is

$$\frac{2p + \mathbf{E}\left[\, e^{-2\int_0^T \lambda ds} \,\right] - 1 - p^2}{p(1-p)} = \frac{\mathbf{E}\left[\, e^{-2\int_0^T \lambda ds} \,\right] - (1-p)^2}{p(1-p)} = \frac{\text{Var}\left(e^{-\int_0^T \lambda ds}\right)}{p(1-p)}$$

which is of order p, because Var $\left(e^{-\int_0^T \lambda ds}\right)$ is of order p^2.

Thus the **maximum** default correlation that can be reached with correlated credit spreads is of the same order of magnitude as the default probabilities.

This is the approach to default correlation taken in CreditRisk+, so it is not surprising that the technical documentation states:

'. . . one would expect default correlations to typically be of the same order of magnitude as default probabilities themselves.'
(CSFP: Credit Risk+ Technical Document p. 57)

i.e. in the order of 0.5 % – 3%.

One might take the differing view that CreditRisk+ is likely to **underestimate** the default correlation and thus the worst–case losses.

Stress Events in Intensity Models

Duffie / Singleton *Simulating Correlated Defaults* (1998)

Instead of correlating default intensities λ_A, λ_B:

- joint default events
  - N'_A with λ'_A: firm A defaults *alone*
  - N'_B with λ'_B: firm B defaults *alone*
  - N'_{AB} with λ'_{AB}: firms A *and* B default *together*
- default of each subportfolio is directly triggered by a jump process of its own

Issues:

- How to distribute the intensity weight over the subsets
- How to reduce the number of subsets

General case: Mixture models

(Frey/McNeil 2001)

- The default probability of an obligor depends on a set of common economic factors.

- Given the factor values, defaults of different obligors are independent.

- Latent variable models such as CreditMetrics and KMV can be interpreted as Bernoulli mixture models, as can McKinsey's CreditPortfolioView.

- CreditRisk$^+$ is essentially a Poisson mixture model.

Thus differences between the models are often related more to presentation and interpretation than to mathematical substance.

The Key: Appropriate modelling of the dependence structure

- To assess default risk in a portfolio, we are in particular interested in *tail dependence*.
- Models such as KMV and CreditMetrics face the problem of potentially underestimating tail dependence.
- Correlations, in particular pairwise correlations, alone do not adequately describe the dependence structure.
- Even when individual default probabilities of obligors *and latent variable correlations* are held fixed, it is still possible to develop alternative models which lead to much heavier–tailed loss distributions. (cf. Frey/McNeil/Nyfeler 2001)

Modelling dependence using copula functions

A joint distribution function can be separated into

- marginal distribution functions of the individual random variables
- a copula function representing the dependence structure

This copula function is unique on the range of the random variables, i.e. the dependence structure is completely characterised by the copula.

Copulas

- A copula is a joint (i.e. multivariate) distribution function with uniform marginals
- Essentially, the decomposition of a multivariate distribution into individual marginal distributions and a copula representing the dependence structure is achieved by rescaling variables using the cumulative distribution functions of the marginals
- If for a bivariate copula $C(\cdot,\cdot)$ we have

$$\lim_{u\to 1}\frac{1+C(u,u)-2u}{1-u}=\eta_U>0 \qquad \text{resp.} \qquad \lim_{u\to 0}\frac{C(u,u)}{u}=\eta_L>0$$

  we say that C has **upper** resp. **lower tail dependence** with parameter η_U resp. η_L.

Copula–dependent default risk in intensity models

(Schönbucher/Schubert 2001)

Use copulas to link classical one–obligor intensity models into a single model.

- for each obligor, intensities can be calibrated to market spreads
- the dependency structure of defaults is given by a copula function
- since it is specified independently of the individual default intensities, any arbitrary dependency structure can be achieved while preserving the calibration to market spreads
- the dependency structure could for example be taken from a portfolio credit risk model of the firm's value type
- there is **default contagion** in the sense that at default events, credit spreads of non-defaulted obligors will jump upwards

The Basic Idea

Linking univariate intensity models via a copula:

- Considered in isolation, the i–th obligor has default intensity λ_i.
- The default time τ_i can be generated via *countdown to default,* i.e.

$$\tau_i = \inf\left\{t \,\middle|\, U_i \geq \exp\left\{-\int_0^t \lambda_i(s)ds\right\}\right\}$$

  where U_i is uniformly distributed on $[0,1]$.

- With the joint distribution function of the U_i (a copula), any desired dependence structure can be implemented.

Default Dynamics

The way that information is revealed drives the dynamics of dependent defaults. Consider two obligors with positive default dependence:

- Initially (at time t_0), there is no particular information.
- Suppose that between t_0 and t_1, there has been no default. This means that U_1 is not that high, so U_2 is likely to be not high either — i.e. given that other obligors have not yet defaulted, an obligor is likely to default in the immediate future.
- Conversely, if obligor 1 defaults at t_1, default of obligor 2 is likely to be not too far off — i.e. the default of one obligor will cause a jump in the conditional default probability of the other obligor.

These **mutual default influences** can be expressed in terms of the copula representing the default dependence structure.

Which copula to use?

(Frey/McNeil/Nyfeler 2001)

- **Example:** Simulation study of Gaussian copula versus Student t copula in a latent variable model
- Homogeneous portfolios, where all default probabilities π are identical and the asset correlation of any two counterparties is the same.
- Three different parameter constellations corresponding to different credit ratings:

Group	π	ρ
A	0.01%	2.58%
B	0.50%	3.80%
C	7.50%	9.21%

Results of Simulation Study

m	Group	$m_{0.95}$				$m_{0.99}$			
		$\nu = \infty$	$\nu = 50$	$\nu = 10$	$\nu = 4$	$\nu = \infty$	$\nu = 50$	$\nu = 10$	$\nu = 4$
1000	A	2	3	3	0	3	6	13	12
1000	B	12	16	24	25	17	28	61	110
1000	C	163	173	209	261	222	241	306	396
10000	A	14	23	24	3	21	49	118	126
10000	B	109	153	239	250	157	261	589	1074
10000	C	1618	1723	2085	2587	2206	2400	3067	3916

Source: Frey/McNeil/Nyfeler (2001)

Drawbacks of Gaussian and Student–t–copulae

Gaussian and Student–t–copulae imply an unrealistic term structure of default dependencies.

E.g. in the Gauss–copula model the dependency approaches ∞ at $t = 0$ and decays strongly as time increases $\Longrightarrow$ the model is strongly date–dependent.

Thus a First–to–default swap priced at $t = 0$ and spanning five years would be much cheaper than an identical swap starting at $t = 1$, priced under identical credit spreads.

Also, these copulae are not very tractable analytically.

Generalised Archimedean Copulae

Rogge and Schönbucher (2003) propose a class of copulae, which do not exhibit the aforementioned disadvantages.

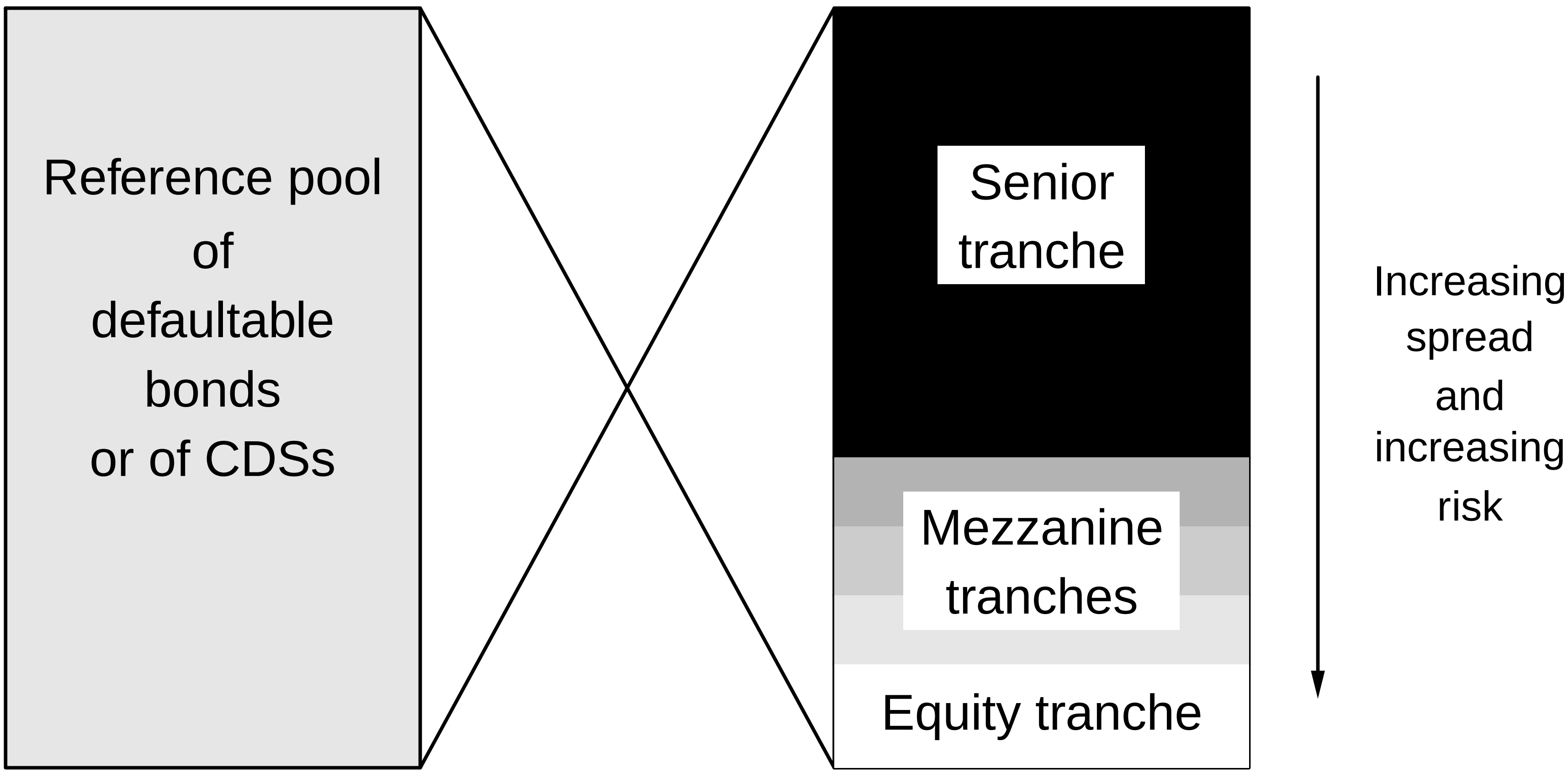
It is a generalisation of the **Archimedean copula function**, which is a copula which can be represented as

$$C(u) = \varphi\left(\sum_{i=1}^{I} \psi(u_i)\right) \qquad \text{where } \psi = \varphi^{[-1]}$$

$\varphi : \mathbf{R}_0^+ \to [0,1]$ is known as the **generator function** of the copula $C(\cdot)$.

- The joint (copula) distribution function is available in closed form even in high–dimensional cases.
- The generation of random variates is straightforward.
- The generalised version also allows some groups (or pairs) of obligors to have a higher default dependency than others.
- The joint relative jump sizes of the hazard rates for every pair of obligors can be explicitly characterised.

CDOs and Synthetic CDOs



Observations

- Pricing CDO tranches is strongly affected by correlation.
- There is a developing market in synthetic tranches defined on standard portfolios.
- For example, dealer quotes have been available since 2003 on five standard TRAC-X tranches:

  0–3%, 3–6%, 6–9%, 9–12%, 12–22% and 10–30%
- Can we speak of **market implied** default correlation on the basis of synthetic tranche spreads?

Dimensionality

- In the most general case, a 100–name CDO would depend on 2^{100} joint default probabilities.
- Assuming a normal dependency structure, we would still need 4950 pair-wise correlations.
- However, only 6 spreads are quoted in the market.
- A radical reduction of dimensionality is necessary.

Several possible ways to reduce the dimensionality of the correlation estimate to 1 have been suggested:*

- Assuming a flat (homogeneous) correlation structure

  $\Longrightarrow$ LHP approximation

- Scaling existing correlations by a constant

- Perturbing eigenvalues of an existing correlation matrix

*cf. Lutz Schlögl and Dominic O'Kane, Assessing Correlation Skew Versus Tail Dependence in CDO Tranche Valuation, presentation at ICBI conference, London, December 2003.

First results* suggest that using a LHP approximation gives a "correlation smile":

1. Fitting the spreads of different tranches (of the same reference portfolio) results in different implied correlations.

2. For the Gaussian copula, the correlation implied by the spreads of the middle tranches is lower than the implied correlation for the lower and upper tranches.

3. Similar effects appear for the Student–t copula.

4. For constant correlation, Gaussian seems to fit better than Student–t, but still not very well.

Results (1) and (2) are also reported by Friend and Rogge (2004).

Conclusions

- The effect of default correlation can be very large (sometimes as large as the default probabilities themselves)
- Default correlation especially important for worst cases and VaR.
- Take care when applying intensity-based models: Spread correlation does not lead to similar default correlation.
- (Pairwise) correlation is an inadequate concept for modelling tail dependence in portfolios — better to use copula functions.
- Good information on default correlation is difficult to come by. However, as standard synthetic CDO tranches become liquidly traded, “implied correlation” may become a viable concept.

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