Market Models of LIBORs and Swap Rates

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HJM APPROACH

1 Heath, Jarrow and Morton Approach

By definition, the instantaneous forward rate $f(t,u)$ satisfies

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right)$$

where $B(t,T)$ is the price at time t of the T-maturity (default-free) discount bond with unit face value.

The Heath, Jarrow and Morton (1992) approach assumes that for any fixed maturity date $T > 0$ the dynamics of the forward rate $f(t,T)$ under the actual probability P are

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW_t$$

where W is a d-dimensional standard Brownian motion (SBM).

1.1 Bond Dynamics

Under the actual (statistical) probability P, the dynamics of the bond price $B(t,T)$, $t \in [0,T]$, are

$$dB(t,T) = B(t,T)\big(a(t,T)\,dt + b(t,T)\,dW_t\big),$$

where the drift coefficient equals

$$a(t,T) = f(t,t) - \alpha^*(t,T) + \tfrac{1}{2}\,|\sigma^*(t,T)|^2$$

and the bond price volatility satisfies

$$b(t,T) = -\sigma^*(t,T) = \int_t^T \sigma(t,u)\,du.$$

Under the spot martingale measure P^* the bond price satisfies

$$dB(t,T) = B(t,T)\big(r_t\,dt + b(t,T)\,dW_t^*\big).$$

1.2 Bond Option

The spot martingale measure P^* is known as the pricing measure. Consider a European call option, with expiry date T, written on a zero-coupon bond of maturity $U > T$. The payoff at expiry equals

$$C_T = (B(T,U) - K)^+$$

and thus the call price C_t at any date $t \le T$ is

$$C_t = B_t \, \mathsf{E}_{\mathsf{P}^*} \left(B_T^{-1} (B(T,U) - K)^+ \,\middle|\, \mathcal{F}_t \right)$$

where the process

$$B_t = \exp\left(\int_0^t r_u \, du \right) = \exp\left(\int_0^t f(u,u) \, du \right)$$

represents the savings account.

1.3 Forward Price

There exists a probability measure P_T such that

$$C_t = B(t,T)\, \mathsf{E}_{\mathsf{P}_T}\big((B(T,U)-K)^+ \,\big|\mathcal{F}_t\big).$$

Let $F(t,U,T) = B(t,U)/B(t,T)$ stand for the forward price at time t of the U-maturity zero-coupon bond for settlement at the date T.

Then clearly

$$C_t = B(t,T)\, \mathsf{E}_{\mathsf{P}_T}\big((F(T,U,T)-K)^+ \,\big|\, \mathcal{F}_t\big).$$

The last formula shows that we need only to know the probability law of the random variable $F(T,U,T)$ under P_T.

We shall now show how to specify P_T given the spot martingale measure P^*.

1.4 Forward Measure

A probability measure P_T on $(\Omega, \mathcal{F}_T)$, equivalent to P^*, with the Radon-Nikodým derivative given by:

$$\frac{d\mathsf{P}_T}{d\mathsf{P}^*} = \frac{B_T^{-1}}{\mathsf{E}_{\mathsf{P}^*}(B_T^{-1})} = \frac{1}{B_T B(0,T)}, \qquad \mathsf{P}^*\text{-a.s.}$$

is the forward martingale measure for the date T.

Notice that

$$\frac{d\mathsf{P}_T}{d\mathsf{P}^*} = \frac{B_0}{B(0,T)} \frac{B(T,T)}{B_T}, \qquad \mathsf{P}^*\text{-a.s.}$$

Hence the martingale measure P_T is associated with the choice of the T-maturity discount bond as a numeraire asset.

1.5 Bond Option Formula

Consider a European call option with expiration date T and strike K. The option is written on a discount bond with maturity $U > T$.

Let C_T be the option's terminal payoff, so that

$$C_T = (B(T,U) - KB(T,T))^+$$

and let C_t stand for the price of this option at time $t \le T$.

For every $t \in [0,T]$, we obtain the generic valuation formula

$$C_t = B(t,U)\mathsf{P}_U(D \,|\, \mathcal{F}_t) - KB(t,T)\mathsf{P}_T(D \,|\, \mathcal{F}_t)$$

where $D = \{B(T,U) > KB(T,T)\}$ is the exercise event of the call option.

1.6 Valuation Formula

Let the bond volatilities $b(t,U)$ and $b(t,T)$ be deterministic. In the Gaussian HJM setup, we have the following pricing result.

Bond Option Formula: The spot price C_t equals

$$C_t = B(t,U)N\big(d_1(t,T)\big) - KB(t,T)N\big(d_2(t,T)\big)$$

where N is the standard normal c.d.f.,

$$d_{1,2}(t,T) = \frac{\ln(B(t,U)/B(t,T)) - \ln K \pm \frac{1}{2}\, v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T |b(u,U) - b(u,T)|^2\, du.$$

1.7 Equivalent Representation

Let $\gamma(t,U,T) = b(t,U) - b(t,T)$ be a deterministic forward volatility. We write $F_C(t,T) = C_t/B(t,T)$ to denote the forward price of a call option.

Bond Option Formula: The forward price $F_C(t,T)$ equals

$$F_C(t,T) = F(t,U,T)N\big(d_1(t,T)\big) - KN\big(d_2(t,T)\big)$$

where

$$d_{1,2}(t,T) = \frac{\ln(F(t,U,T)) - \ln K \pm \frac{1}{2}\,v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T |\gamma(u,U,T)|^2\,du.$$

GENERAL SET-UP

2 General Set-up

Consider two particular portfolios of discount bonds, with the wealth processes V_t^1 and V_t^2. A term structure model is not explicitly specified here.

We are interested in an option to exchange one of these portfolios for another, at a given exercise date T. Let us write

$$C_T = (V_T^1 - KV_T^2)^+.$$

This means that

$$C_T = V_T^1 \, \mathbb{1}_D - KV_T^2 \, \mathbb{1}_D$$

where $K > 0$ is a constant strike, and $D = \{V_T^1 > KV_T^2\}$ stands for the exercise event.

2.1 Generic Valuation Formula

Using the abstract Bayes rule, it is not difficult to check that the equality

$$\frac{d\mathsf{P}^1}{d\mathsf{P}^2} = \frac{V_0^2}{V_0^1}\,\frac{V_T^1}{V_T^2}, \qquad \mathsf{P}^2\text{-a.s.}$$

provides a link between the martingale measures P^1 and P^2 associated with the choice of wealth processes V^1 and V^2 as numeraires. Both probability measures are given on $(\Omega, \mathcal{F}_T)$.

The price of a call option admits the following representation, for $t \in [0, T]$,

$$C_t = V_t^1 \mathsf{P}^1(D \,|\, \mathcal{F}_t) - K V_t^2 \mathsf{P}^2(D \,|\, \mathcal{F}_t)$$

where $D = \{V_T^1 > K V_T^2\}$ is the exercise set.

2.2 Lognormality of Relative Prices

To obtain the Black formula for the option price C_t, it is enough to make the following assumption.

Lognormality Assumption: The relative price V_t^1/V_t^2 follows a lognormal martingale under P^2. More specifically,

$$d\,(V_t^1/V_t^2) = (V_t^1/V_t^2)\,\gamma_t\,dW_t^2$$

for a deterministic function $\gamma : [0,T] \to \mathrm{R}^d$. The process W^2 is assumed to follow a SBM under the probability measure P^2.

Technical Assumption: We also assume that the function γ is such that bounded

$$\mathrm{P}^2 \left(\int_0^T |\gamma_t|^2\,dt < \infty \right) = 1.$$

2.3 Volatility Invariance

Girsanov Theorem: The Radon-Nikodým density of P^1 with respect to P^2 equals

$$\frac{d\mathsf{P}^1}{d\mathsf{P}^2} = \exp\left(\int_0^T \gamma_t \, dW_t^2 - \frac{1}{2}\int_0^T |\gamma_t|^2 \, dt\right) \qquad \mathsf{P}^2\text{-a.s.}$$

and thus the process

$$W_t^1 = W_t^2 - \int_0^t \gamma_u \, du, \quad \forall \, t \in [0, T],$$

is a standard Brownian motion under P^1. It appears that

$$d\,(V_t^2/V_t^1) = -(V_t^2/V_t^1)\,\gamma_t \, dW_t^1.$$

2.4 Generic Formula in Lognormal Models

Reasoning in the same way as in the proof of the classic Black-Scholes formula, we obtain the following result.

Generic Formula: The price C_t equals

$$C_t = V_t^1 N\big(d_1(t,T)\big) - K V_t^2 N\big(d_2(t,T)\big)$$

where

$$d_{1,2}(t,T) = \frac{\ln(V_t^1/V_t^2) - \ln K \pm \frac{1}{2}\, v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T |\gamma_u|^2 \, du, \quad \forall\, t \in [0,T].$$

A cap (a swaption) valuation formula in the lognormal market model of forward LIBORs (forward swap rates) is a special case of this generic result.

2.5 Special Cases

For the j^{th} caplet, we take:

$$V_t^1 = B(t, T_j) - B(t, T_{j+1}), \quad V_t^2 = \delta_j B(t, T_{j+1}).$$

For the j^{th} co-terminal swaption, we take:

$$V_t^1 = B(t, T_j) - B(t, T_{n+1}), \quad V_t^2 = \sum_{i=j+1}^{n+1} \delta_{i-1} B(t, T_i).$$

To derive the generic valuation formula for $t = 0$, it suffices to assume that the random variable V_T^1 / V_T^2 has a lognormal probability law under the martingale measure P^2. This observation underpins the construction of the so-called Markov-functional interest rate models.

MARKET MODELS OF LIBORs

3 Market Models of LIBORs

The forward LIBOR $L(t,T)$, prevailing at time $t \leq T$ for the accrual period $[T, T+\delta]$, is given by the formula

$$1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F(t,T,T+\delta).$$

Equivalently,

$$L(t,T) = \frac{B(t,T) - B(t,T+\delta)}{\delta B(t,T+\delta)}\,.$$

It is clear that $L(t,T) > 0$ whenever $B(t,T) > B(t,T+\delta)$. The latter condition is not obvious in the Gaussian HJM set-up.

3.1 Dynamics of the Forward LIBOR: HJM Set-up

We have, under the forward martingale measure $\mathsf{P}_{T+\delta}$,

$$dL(t,T) = \delta^{-1} F(t,T,T+\delta)\, \gamma(t,T,T+\delta)\, dW_t^{T+\delta}$$

where $W^{T+\delta}$ is an SBM under $\mathsf{P}_{T+\delta}$ and

$$\gamma(t,T,T+\delta) = b(t,T) - b(t,T+\delta).$$

Put another way

$$dL(t,T) = \delta^{-1}\big(1 + \delta L(t,T)\big)\, \gamma(t,T,T+\delta)\, dW_t^{T+\delta}.$$

3.2 Volatility of Forward LIBOR

Let us write

$$dL(t,T) = L(t,T)\,\lambda(t,T)\,dW_t^{T+\delta}$$

where we set

$$\lambda(t,T) = \frac{1+\delta L(t,T)}{\delta L(t,T)}\,\gamma(t,T,T+\delta).$$

This shows that the volatility $\lambda(t,T)$ of the forward LIBOR is stochastic when the bond volatilities $b(t,T)$ and $b(t,T+\delta)$ are deterministic.

We conclude that the lognormality of forward LIBOR is not consistent with the Gaussian HJM set-up.

3.3 Caps and Floors

An interest rate cap is a portfolio of call options on (spot) LIBOR.

A particular caplet maturing at time T pays at the settlement date $T + \delta$ the amount

$$\mathsf{Cpl}_{T+\delta} = (L(T) - \kappa)^{+} \delta P$$

where $L(T) = L(T, T)$ is the spot LIBOR, $\kappa > 0$ is a fixed level of interest rate and P is the nominal principal. Without loss of generality, we set $P = 1$.

Formally, a caplet (floorlet) is equivalent to a put (call) option written on a discount bond maturing at T. This is clear, since

$$B(T, T + \delta)\, \mathsf{Cpl}_{T+\delta} = (K\delta)^{-1} \big(K - B(T, T + \delta)\big)^{+}$$

where we write $K = (1 + \kappa\delta)^{-1}$.

3.4 Market Valuation Formula

Consider a caplet with expiry date T, settlement date $T + \delta$, and strike level κ. Recall that the claim

$$C_T = \big(L(T) - \kappa\big)^+ \delta = \big(L(T,T) - \kappa\big)^+ \delta$$

is settled at time $T + \delta$.

Market practice is to price a caplet assuming that the underlying forward interest rate process is lognormally distributed. Assume that the forward LIBOR $L(t,T)$ satisfies under Q

$$dL(t,T) = L(t,T)\sigma \, dW_t$$

where W is a one-dimensional SBM and $\sigma > 0$ is a constant. We thus have

$$L(t,T) = L(0,T) \, e^{\,\sigma W_t - \frac{1}{2}\sigma^2 t^2}.$$

3.5 Black Formula for Caplets

By convention, the market price at time t of a caplet equals

$$C_t = \delta B(t, T+\delta)\Big(L(t,T)N\big(d_1(t,T)\big) - \kappa N\big(d_2(t,T)\big)\Big)$$

where N is the standard Gaussian cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz$$

and $d_{1,2}(t,T)$ satisfy

$$d_{1,2}(t,T) = \frac{\ln(L(t,T)/\kappa) \pm \frac{1}{2}\,\sigma^2(T-t)}{\sigma(T-t)}.$$

The constant σ is the implied volatility of the caplet.

3.6 MSS Approach

Miltersen, Sandmann and Sondermann (1997) postulated that the forward LIBOR $L(\cdot, T)$ satisfies, under the spot martingale measure P^*,

$$dL(t,T) = \mu(t,T)\,dt + L(t,T)\lambda(t,T)\,dW_t^*$$

with a deterministic volatility function $\lambda(\cdot, T)$.

The forward price $F_t = F(t, T+\delta, T)$ of a discount bond thus satisfies

$$dF_t = -F_t\big(1 - F_t\big)\lambda(t,T)\,dW_t^T.$$

The SDE above has a unique solution for any initial condition $0 < F_0 < 1$.

3.7 PDE Approach

MSS (1997) focused on the PDE satisfied by the function $v = v(t, x)$ yielding the forward price of the put bond option in terms of the forward bond price.

The PDE for the put bond option price is

$$\frac{\partial v}{\partial t} + \frac{1}{2} |\lambda(t, T)|^2 x^2 (1 - x)^2 \frac{\partial^2 v}{\partial x^2} = 0$$

with the terminal condition, $v(T, x) = (K - x)^+$.

Using the PDE above, MSS (1997) derived a closed-form expression for the price of the European put option written on a discount bond, and they formally established the market formula for caplets.

3.8 Black Caplet Formula

Let us state the main result of MSS (1997).

MSS Formula: The price at time $t \in [0,T]$ of a caplet with strike rate κ, maturing at T, is given by the market formula

$$\mathrm{Cpl}_t = \delta B(t, T+\delta)\,\big(L(t,T)N\big(d_1(t,T)\big) - \kappa\, N\big(d_2(t,T)\big)\big)$$

where

$$d_{1,2}(t,T) = \frac{\ln(L(t,T)/\kappa) \pm \frac{1}{2}\, v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T |\lambda(u,T)|^2\, du.$$

3.9 BGM Approach

To provide a formal construction of a lognormal model of forward LIBORs, Brace, Gațarek and Musiela (1997) start working within the HJM set-up.

They postulate that the dynamics of $L(t,T)$ under the spot martingale measure P* are

$$dL(t,T) = \mu(t,T)\,dt + L(t,T)\lambda(t,T)\,dW_t^*$$

where $\lambda(t,T)$ is deterministic and the drift coefficient $\mu(t,T)$ is unspecified.

The BGM approach hinges on a judicious choice of the drift $\mu(t,T)$.

3.10 Auxiliary Relationship

The instantaneous forward rate $f(t, T)$ satisfies under P^*

$$df(t,T) = \sigma(t,T)\,\sigma^*(t,T)\,dt + \sigma(t,T)\,dW_t^*.$$

On the other hand

$$1 + \delta L(t,T) = \exp\Big(\int_T^{T+\delta} f(t,u)\,du\Big).$$

Applying Itô's formula to both sides of the last equality, and comparing the diffusion terms, we obtain

$$b(t,T) - b(t,T+\delta) = \frac{\delta L(t,T)}{1+\delta L(t,T)}\,\lambda(t,T).$$

3.11 Additional Assumption

To solve the last equation for b in terms of L, one needs to impose an initial condition.

For instance, by setting $\sigma(t,T)=0$ for every $t \in [T-\delta, T]$, we obtain

$$b(t,T) = -\sum_{k=1}^{[\delta^{-1}(T-t)]} \frac{\delta L(t,T-k\delta)}{1+\delta L(t,T-k\delta)}\,\lambda(t,T-k\delta).$$

This particular choice was made by Brace, Gątarek and Musiela (1997).

Then the process $L(t,T)$ satisfies under the spot martingale measure $\mathbb{P}^*$

$$dL(t,T) = L(t,T)\big(-b(t,T)\,\lambda(t,T)\,dt + \lambda(t,T)\,dW_t^*\big)$$

with the coefficient $b(t,T)$ as given above.

3.12 BGM Model

Existence and uniqueness of solutions to the SDE that governs the forward LIBOR $L(t,T)$ for the drift b can be established using the forward induction.

BGM (1997) proved the existence a consistent family of forward LIBORs such that each process $L(\cdot,T)$ is lognormally distributed under the forward martingale measure $\mathsf{P}_{T+\delta}$

$$dL(t,T) = L(t,T)\lambda(t,T)\, dW_t^{T+\delta}.$$

Essential features of the BGM model:

(a) the length δ of the accrual period is fixed,

(b) the maturity T of forward LIBOR is any future date.

3.13 Comments on BGM Model

Remark 1. The assumption that $\sigma(t,T)=0$ for $t \in [T-\delta, T]$ implies that there is no randomness in the bond price $B(t,T)$ for every $t \in [T-\delta, T]$, that is, when the bond is close to maturity.

Remark 2. The assumption that the length of each accrual period is constant does not fit well market practice. In practice, it is essential to consider accrual periods of differing lengths.

Remark 3. Although BGM approach was based on the HJM set-up, the two frameworks are in fact inconsistent.

3.14 Discrete Tenor Structure

We now consider a tenor structure $0 < T_1 < T_2 < \cdots < T_{n+1}$. We denote by $\delta_j = T_{j+1} - T_j$ the length of the jth accrual period for $j = 1, 2, \ldots, n$.

The forward LIBOR $L(t, T_j)$ is set to satisfy, for every $t \in [0, T_j]$,

$$1 + \delta_j L(t, T_j) = \frac{B(t, T_j)}{B(t, T_{j+1})}.$$

Note that

$$1 + \delta_j L(t, T_j) = \mathsf{E}_{\mathsf{P}_{T_{j+1}}}(B^{-1}(T_j, T_{j+1}) \,|\, \mathcal{F}_t).$$

Hence, the forward LIBOR $L(t, T_j)$ follows a martingale under the forward martingale measure $\mathsf{P}_{T_{j+1}}$ associated with the end of the accrual period.

3.15 Modelling of LIBORs

Assume that each LIBOR $L_i(t) = L(t, T_i)$, $i = 1, 2, \dots, n$, solves under the statistical probability P the following stochastic differential equation (SDE)

$$dL_i(t) = L_i(t)\big(\mu_i(t)\,dt + \sigma_i(L_i(t), t)\,dW_t^i\big).$$

The drift coefficient $\mu_i(t)$ depends on forward LIBORs L_j, $j = 1, 2, \dots, n$, existing at time t, and is sufficiently regular to ensure the existence and uniqueness of a solution to this SDE.
One-dimensional Brownian motions $W^1, W^2, \dots, W^n$ are correlated, and their instantaneous correlations under P are given by

$$d\langle W^i, W^j\rangle_t = \rho_{i,j}(t)\,dt.$$

3.16 Drift Term under $\mathsf{P}_{T_{n+1}}$

The drift term $\widehat{\mu}_i(t)$ in the dynamics of $L_i(t) = L(t, T_i)$ under the forward measure $\mathsf{P}_{T_{n+1}}$ equals

$$\widehat{\mu}_i(t) = -\sum_{j=i+1}^{n} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)} \sigma_i(L_i(t), t)\sigma_j(L_j(t), t)\rho_{i,j}(t).$$

The SDE for $L_1, L_2, \dots, L_n$ under the forward measure $\mathsf{P}_{T_{n+1}}$ has the form

$$dL_i(t) = L_i(t)\left(-\sum_{j=i+1}^{n} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)} \sigma_i(L_i, t)\sigma_j(L_j, t)\rho_{i,j}(t)\, dt + \sigma_i(L_i, t)\, d\widehat{W}^i_t\right)$$

where $\widehat{W}^1, \widehat{W}^2, \dots, \widehat{W}^n$ are Brownian motions with instantaneous correlations

$$d\langle \widehat{W}^i, \widehat{W}^j \rangle_t = d\langle W^i, W^j \rangle_t = \rho_{i,j}(t)\, dt$$

3.17 Valuation of Caps in LLM

We shall now examine the valuation of caps within the lognormal model of forward LIBORs (LLM). Here, we set $\sigma_j(L_j(t), t) = \lambda(t, T_j)$.

For every $j = 1, 2, \ldots, n$, the dynamics of the forward LIBOR $L(t, T_j)$ under the forward probability measure $\mathrm{P}_{T_{j+1}}$ are

$$dL(t, T_j) = L(t, T_j)\, \lambda(t, T_j)\, dW_t^{T_{j+1}}$$

where $W^{T_{j+1}}$ follows a d-dimensional SBM under the forward measure $\mathrm{P}_{T_{j+1}}$.

The LIBOR volatility $\lambda(\cdot, T_j) : [0, T_j] \to \mathrm{R}^d$ is a deterministic function.

3.18 Caps Valuation Formula in LLM

Consider an interest rate cap with strike level κ, settled in arrears at times T_j, $j = 2, 3, \ldots, n+1$. We assume the lognormal market model of forward LIBORs.

Formula for Caps: The price of a cap at time $t \in [0, T_1]$ equals

$$\mathsf{FC}_t = \sum_{j=1}^{n} \delta_j B(t, T_{j+1}) \left(L(t, T_j) N\big(d_1(t, T_j)\big) - \kappa N\big(d_2(t, T_j)\big)\right)$$

where

$$d_{1,2}(t, T_j) = \frac{\ln(L(t, T_j)/\kappa) \pm \frac{1}{2} v^2(t, T_j)}{v(t, T_j)}$$

and

$$v^2(t, T_j) = \int_t^{T_j} |\lambda(u, T_j)|^2 \, du.$$

3.19 Jamshidian's Approach

Jamshidian (1997) developed an alternative approach to modelling of LIBORs by introducing a specific numeraire asset.

A spot LIBOR measure is a probability measure under which all relative bond prices are local martingales, when the price process obtained by rolling over one-period bonds is taken as a numeraire asset.

Let us set, for every $t > 0$,

$$G_t = B(t, T_{m(t)}) \prod_{j=1}^{m(t)} B^{-1}(T_{j-1}, T_j)$$

where for every $t > 0$

$$m(t) = \inf \{ k \in \mathrm{N} \,|\, T_k \geq t \}.$$

3.20 Spot LIBOR Measure

Note that G_t represents the wealth at time t of a portfolio which starts at time 0 with one unit of cash invested in a discount bond of maturity T_1. At each reset date T_j, $j = 1, 2, \ldots, n$, the wealth is then reinvested in discount bond maturing at the next date; that is, at time T_{j+1}.

The process G can be interpreted as a proxy of a savings account.

We choose the LIBOR savings account G as a numeraire asset, and we introduce the corresponding martingale measure.

A spot LIBOR measure P^L is a probability measure to P, such that for every $j = 1, 2, \ldots, n+1$ the relative bond price $B(t, T_j)/G_t$ follows a martingale under P^L.

3.21 Jamshidian's SDE: General Case

Jamshidian's version of a LIBOR model is formally given as a solution to a multidimensional stochastic differential equation.

For any $j = 1, 2, \ldots, n$, the forward LIBOR $L(\cdot, T_j)$ satisfies

$$dL(t, T_j) = \sum_{i=m(t)}^{j} \frac{\delta_i \zeta(t, T_i) \zeta(t, T_j)}{1 + \delta_i L(t, T_i)} \, dt + \zeta(t, T_j) \, dW_t^L$$

for some $\mathbb{R}^d$-valued processes $\zeta(t, T_j)$, where W^L is a d-dimensional SBM under the spot LIBOR measure $\mathbb{P}^L$.

3.22 Jamshidian's SDE: Special Case

The SDE for the forward LIBORs under the spot LIBOR measure $\mathbb{P}^L$ can be represented as follows

$$dL_j(t) = L_j(t) \left(\sum_{i=m(t)}^{j} \frac{\delta_i L_i(t)}{1 + \delta_i L_i(t)} \sigma_i(L_i, t) \sigma_j(L_j, t) \rho_{i,j}(t)\, dt + \sigma_j(L_j, t)\, d\widetilde{W}_t^j \right)$$

where $\widetilde{W}^1, \widetilde{W}^2, \ldots, \widetilde{W}^n$ are one-dimensional correlated Brownian motions under $\mathbb{P}^L$ with instantaneous correlations

$$d\langle \widetilde{W}^i, \widetilde{W}^j \rangle_t = d\langle W^i, W^j \rangle_t = \rho_{i,j}(t)\, dt$$

for every $1 \leq i, j \leq n$.

3.23 CEV LIBOR Model

Anderson and Andreasen (2000) examine a variant of a model of forward LIBOR rates in which

$$\zeta(t, T_j) = \lambda(t, T_j) L^{\alpha}(t, T_j)$$

for some constant $\alpha \geq 1/2$. The resulting CEV (constant elasticity of variance) model was previously examined in the context of equity derivatives by Cox and Ross (1976). It is known that for any $\alpha \geq 1/2$ the SDE

$$dL(t, T_j) = \lambda(t, T_j) L^{\alpha}(t, T_j) \, dW_t^{T_{j+1}}$$

admits a unique strong solution, and thus the model of forward LIBORs can be easily constructed in this case. The case of $\alpha < 1/2$ is more difficult to handle.

FORWARD SWAP RATES

4 Co-Terminal Forward Swap Rates

Consider a forward-start fixed-for-floating interest rate swap settled in arrears, with notional principal P.

We introduce a finite collection of dates $0 < T_1 < \cdots < T_{n+1}$ and we denote $\delta_j = T_{j+1} - T_j > 0$ for every $j = 1, 2, \ldots, n+1$.

The floating rate $L(T_j)$ received at time T_{j+1} is set at time T_j and it equals the spot LIBOR prevailing at time T_j, so that

$$L(T_j) = \frac{1 - B(T_j, T_{j+1})}{\delta_j B(T_j, T_{j+1})}.$$

4.1 Payer Swap

At any date $T_{j+1},\ j = 1, \ldots, n,$ the cash flows of a forward payer swap are

$$L(T_j)\delta_j P \quad \text{and} \quad -\kappa\delta_j P,$$

where κ is a preassigned fixed rate of interest.

4.2 Receiver Swap

At any date $T_{j+1},\ j = 1, \ldots, n,$ the cash flows of a forward receiver swap are

$$-L(T_j)\delta_j P \quad \text{and} \quad \kappa\delta_j P.$$

A short position in a payer swap is equivalent to a long position in a receiver swap.

4.3 Value of a Payer Swap: First Formula

From now on, we shall set $P = 1$. A simple arbitrage argument shows that the value at time t of a forward payer swap, denoted by $\mathsf{FS}_t(\kappa)$, equals

$$\mathsf{FS}_t(\kappa) = B(t, T_1) - \sum_{j=2}^{n+1} c_j B(t, T_j)$$

for every $t \in [0, T_1]$, where the coupons c_j are:

$$c_j = \kappa \delta_{j-1}$$

for $j = 2, \ldots, n$, and $c_{n+1} = 1 + \kappa \delta_n$.

In essence, a forward payer swap settled in arrears is a contract to deliver a specific coupon-bearing bond and to receive in the same time a particular discount bond.

4.4 Forward Swap Rate

Let us write $T = T_1$. The forward swap rate $\kappa(t, T, n)$, at time t is that value of the fixed rate κ for which the forward swap is valueless, that is, $\text{FS}_t(\kappa) = 0$.

Simple calculations show that

$$\kappa(t, T, n) = \frac{B(t, T) - B(t, T_{n+1})}{\sum_{j=2}^{n+1} \delta_{j-1} B(t, T_j)} .$$

Swap rate is the forward swap rate, with $t = T$.

The swap rate $\kappa(T, T, n)$ equals

$$\kappa(T, T, n) = \frac{1 - B(T, T_{n+1})}{\sum_{j=2}^{n+1} \delta_{j-1} B(T, T_j)} .$$

4.5 Value of a Payer Swap: Second Formula

In view of the definition of the forward swap rate $\kappa(t,T,n)$, it is clear that

$$\mathsf{FS}_t(\kappa(t,T,n)) = 0$$

for any date $t \in [0,T]$.

Therefore

$$\mathsf{FS}_t(\kappa) = \mathsf{FS}_t(\kappa) - \mathsf{FS}_t(\kappa(t,T,n))$$

and thus we obtain the following formula for the value of the payer swap

$$\mathsf{FS}_t(\kappa) = \sum_{j=2}^{n+1} \delta_{j-1} B(t,T_j)\big(\kappa(t,T,n) - \kappa\big).$$

In particular, the inequality $\mathsf{FS}_T(\kappa) > 0$ holds if and only if $\kappa(T,T,n) > \kappa$.

4.6 Payer and Receiver Swaptions

The owner of a (payer) swaption with strike rate κ, maturing at time $T = T_1$, has the right to enter at time T the underlying forward payer swap settled in arrears. This means that the payoff at time T equals: $\mathrm{PS}_T = \mathrm{FS}_T(\kappa)$ if $\mathrm{FS}_T(\kappa) > 0$, and $\mathrm{PS}_T = 0$ otherwise.

The swaption is essentially equivalent a sequence of payments

$$s_j = (\kappa(T, T, n) - \kappa)^+ \delta_{j-1}$$

that are received at settlement dates $T_2, T_3, \ldots, T_{n+1}$.

The swaption can thus be seen a specific call option on a forward swap rate, with fixed strike level κ.

4.7 Black Formula for Swaptions

The commonly used formula for pricing (quoting) a swaption in terms of the (implied) volatility is the Black swaptions formula:

$$\mathrm{PS}_t = G_t(n)\left(\kappa(t,T,n)N\big(h_1(t,T)\big) - \kappa N\big(h_2(t,T)\big)\right)$$

where the process $G(n)$ equals

$$G_t(n) = \sum_{j=2}^{n+1} \delta_{j-1} B(t,T_j)$$

and

$$h_{1,2}(t,T) = \frac{\ln(\kappa(t,T,n)/\kappa) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

for some constant $\sigma > 0$. This formula is not based on any specific model, however. It should thus be seen as a market convention.

4.8 Co-Terminal Swaps

For a finite collection of dates $0 < T_1 < \cdots < T_{n+1}$, we consider a particular family of forward start swaps with the end date T_{n+1}.

The floating rate received at time T_{j+1} is the spot LIBOR prevailing at time T_j

$$L(T_j) = L(T_j, T_j) = \frac{1 - B(T_j, T_{j+1})}{\delta_{j+1} B(T_j, T_{j+1})}\,.$$

If a swap starts at T_j and ends at T_{n+1} then its value at time $t \le T_j$ is

$$\mathsf{FS}_t^{\,j}(\kappa) = B(t, T_j) - \sum_{k=j+1}^{n+1} c_k B(t, T_k)$$

where $c_k = \kappa\delta_{k-1}$ for $k = 2, \dots, n$, and $c_{n+1} = 1 + \kappa\delta_n$.

4.9 Co-terminal Forward Swap Rates

The forward swap rate $\kappa(t, T_j, n+1-j)$ is given by the formula

$$\kappa(t, T_j, n+1-j) = \frac{B(t, T_j) - B(t, T_{n+1})}{\delta_j B(t, T_{j+1}) + \cdots + \delta_n B(t, T_{n+1})}$$

for every $t \in [0, T_j]$, $j = 1, 2, \ldots, n$.

We wish to model the family of forward swap rates $\kappa(t, T_j, n+1-j)$ for $j = 1, 2, \ldots, n$.

It is clear that the underlying swaps differ in length (that is, the number of accrual periods), but they all have a common end date. We shall use the term co-terminal forward swap rates.

4.10 Jamshidian's Model

Jamshidian (1997) constructs a family

$$\widetilde{\kappa}(t, T_j) = \kappa(t, T_j, n + 1 - j), \quad j = 1, 2, \dots, n,$$

of forward swap rates, a set of equivalent probability measures $\widetilde{\mathsf{P}}_{T_j}$ and a family $\widetilde{W}^{T_j}$ of processes.

He postulates that that:
(i) for any $j = 2, \dots, n + 1$ the process $\widetilde{W}^{T_j}$ follows an SBM under $\widetilde{\mathsf{P}}_{T_j}$,
(ii) for any $j = 1, \dots, n$ the process $\widetilde{\kappa}(t, T_j)$ satisfies, for $t \in [0, T_j]$,

$$d\widetilde{\kappa}(t, T_j) = \widetilde{\kappa}(t, T_j)\, \nu(t, T_j)\, d\widetilde{W}_t^{T_{j+1}}$$

with the initial condition $\widetilde{\kappa}(0, T_j) > 0$,
(iii) the processes (or functions) $\nu(t, T_j)$ are given in advance.

4.11 Level Process

For any $j = 1, \ldots, n$ we introduce the level process $\widetilde{G}_t(j)$ by setting

$$\widetilde{G}_t(j) = G_t(n+1-j) = \sum_{i=j+1}^{n+1} \delta_{i-1} B(t, T_i).$$

This process is also known as PVBP (the present value of basis point).

For a fixed $j = 1, \ldots, n$, a probability $\widetilde{\mathsf{P}}_{T_{j+1}}$, equivalent to P, is called the forward swap measure for the date T_{j+1} if the relative bond price

$$\frac{B(t, T_k)}{\widetilde{G}_t(j)} = \frac{B(t, T_k)}{\delta_j B(t, T_{j+1}) + \cdots + \delta_n B(t, T_{n+1})}$$

follows a martingale under $\widetilde{\mathsf{P}}_{T_{j+1}}$ for every $k = 1, \ldots, n+1$. Note that

$$\widetilde{\kappa}(t, T_j) = \frac{B(t, T_j) - B(t, T_{n+1})}{\widetilde{G}_t(j)}.$$

4.12 Recursive Relationship

Recall that $\widetilde{\kappa}(t, T_j) = \kappa(t, T_j, n+1-j)$. Let us set for $1 \le j \le i \le n+1$

$$g_t^{ij} = \sum_{k=i}^{n} \delta_{k+1} \prod_{m=j+1}^{k} (1 + \delta_m \widetilde{\kappa}(t, T_m))$$

One can show (by induction) that $\widetilde{G}_t(j) = B(t, T_{n+1}) g_t^{jj}$.

Observe that $\widetilde{\mathsf{P}}_{T_{n+1}} = \mathsf{P}_{T_{n+1}}$ is the forward measure for the date T_{n+1}.

We may thus set $\widetilde{W}^{T_{n+1}} = \widetilde{W}$ for some SBM $\widetilde{W}$.

If for every j

$$d\widetilde{\kappa}(t, T_j) = \mu_t^j \, dt + \phi_t^j \, d\widetilde{W}_t$$

then we obtain the following recursive relationship

$$d\widetilde{\kappa}(t, T_j) = -\sum_{k=j+1}^{n} \frac{\delta_k g_t^{jk} \phi_t^j \phi_t^k}{(1 + \delta_k \widetilde{\kappa}(t, T_k)) g_t^{jj}} \, dt + \phi_t^j \, d\widetilde{W}_t.$$

4.13 Lognormal Model of Co-Terminal Swap Rates

Assume that

$$d\widetilde{\kappa}(t,T_j) = \mu_t^j\,dt + \nu(t,T_j)\widetilde{\kappa}(t,T_j)\,d\widetilde{W}_t$$

Using the recursive relationship, we obtain explicit formulae

$$\widetilde{\kappa}(t,T_n) = \widetilde{\kappa}(0,T_n)\,\exp\left(\int_0^t \nu(u,T_n)\,d\widetilde{W}_u - \frac{1}{2}\int_0^t |\nu(u,T_n)|^2\,du\right)$$

and for every $j = 1,\dots,n-1$

$$\widetilde{\kappa}(t,T_j) = \widetilde{\kappa}(0,T_j)h_t^j\,\exp\left(\int_0^t \nu(u,T_j)\,d\widetilde{W}_u - \frac{1}{2}\int_0^t |\nu(u,T_j)|^2\,du\right)$$

where we write

$$h_t^j = \exp\left(-\int_0^t \sum_{i=j+1}^n \frac{\delta_i g_u^{ji}\nu(u,T_j)\nu(u,T_i)\widetilde{\kappa}(u,T_i)}{(1+\delta_i\widetilde{\kappa}(u,T_i))g_u^{jj}}\,du\right).$$

4.14 Valuation of Co-Terminal Swaptions

For a fixed date T_j, $j = 1, 2, \ldots, n,$ consider a swaption with the expiration date $T_j,$ written on a payer swap with fixed rate κ, which starts at date T_j and has $n + 1 - j$ accrual periods.

The j^{th} swaption can be seen as a contract which pays to its holder the amount

$$\delta_k(\widetilde{\kappa}(T_j, T_j) - \kappa)^+$$

at each settlement date T_k, where $k = j + 1, j + 2, \ldots, n + 1.$

Equivalently, it pays at its maturity date T_j the following lump amount

$$Y = \widetilde{G}_{T_j}(j)\big(\widetilde{\kappa}(T_j, T_j) - \kappa\big)^+,$$

where, as before, $\widetilde{G}_t(j) = \sum_{k=j+1}^{n+1} \delta_{k-1} B(t, T_k).$

4.15 Jamshidian's Formula

The following valuation result was established by Jamshidian (1997). Recall that we denote $\widetilde{\kappa}(t, T_j) = \kappa(t, T_j, n-j)$ for $j = 1, 2, \ldots, n$.

Formula for Co-Terminal Swaptions: The arbitrage price of the j^{th} swaption equals, for $t \in [0, T_j]$,

$$\mathsf{PS}_t^j = \widetilde{G}_t(j) \ \big(\widetilde{\kappa}(t, T_j) N\big(h_1(t, T_j)\big) - \kappa N\big(h_2(t, T_j)\big)\big)$$

where

$$h_{1,2}(t, T_j) = \frac{\ln(\widetilde{\kappa}(t, T_j)/\kappa) \pm \frac{1}{2}\, v^2(t, T_j)}{v(t, T_j)}$$

with

$$v^2(t, T_j) = \int_t^{T_j} |\nu(u, T_j)|^2 \, du.$$

4.16 Hedging of Swaptions

A replicating strategy for a swaption in Jamshidian's model has similar features as a replicating strategy for a caplet in the lognormal model of forward LIBORs.

Let us fix j, and let us denote by $F_{S^j}(t,T)$ the relative price at time $t \leq T_j$ of the j^{th} swaption, when the level process

$$\widetilde{G}_t(j) = \sum_{i=j+1}^{n+1} \delta_{i-1} B(t,T_i)$$

is chosen as a numeraire asset. To be more specific, we set

$$F_{S^j}(t,T_j) = \frac{\mathsf{PS}_t^j}{\widetilde{G}_t(j)}.$$

4.17 Auxiliary Lemma

We find easily that for every $t \leq T_j$

$$F_{S^j}(t, T_j) = \widetilde{\kappa}(t, T_j) N\big(h_1(t, T_j)\big) - \kappa N\big(h_2(t, T_j)\big).$$

Applying Itô's formula to the last expression, we obtain the following useful result.

Dynamics of $F_{S^j}(t, T)$. The dynamics of the process $F_{S^j}(t, T)$ under the forward swap measure $\widetilde{\mathsf{P}}_{T_{j+1}}$ are

$$dF_{S^j}(t, T_j) = N\big(h_1(t, T_j)\big)\, d\widetilde{\kappa}(t, T_j)$$

where

$$h_1(t, T_j) = \frac{\ln(\widetilde{\kappa}(t, T_j)/\kappa) + \frac{1}{2}\, v^2(t, T_j)}{v(t, T_j)}.$$

4.18 Hedging Strategy

For a fixed j, we deal with the following self-financing hedging strategy for the j^{th} swaption.

At time $t = 0$. We start our trade at time 0 with the amount PS_0^j of cash (the premium received from the buyer), which is then immediately invested in the portfolio $\widetilde{G}(j)$. One unit of $\widetilde{G}(j)$ costs $\sum_{k=j+1}^{n+1} \delta_{k-1} B(0, T_k)$ at time 0.

At time $t > 0$. At any time $t \leq T_j$ we assume $\psi_t^j = N\big(h_1(t, T_j)\big)$ positions in market forward swaps (of course, these swaps have the same starting date and tenor structure as the underlying forward swap).

We need to evaluate profits/losses from continuous trading in forward swaps.

4.19 Replication

The associated profits/losses process V, when expressed in units of the level process $\widetilde{G}(j)$, satisfies

$$dV_t = \psi_t^j \, d\widetilde{\kappa}(t, T_j) = N\big(h_1(t, T_j)\big) \, d\widetilde{\kappa}(t, T_j) = dF_{S^j}(t, T_j)$$

with $V_0 = 0$.

Consequently,

$$\mathrm{PS}\,^j_{T_j} = F_{S^j}(T_j, T_j) = F_{S^j}(0, T_j) + \int_0^{T_j} \psi_t^j \, d\widetilde{\kappa}(t, T_j) = F_{S^j}(0, T_j) + V_{T_j}.$$

The last equality makes it clear that the strategy ψ^j introduced above replicates the j^{th} swaption.

Dynamic trading in market forward swap occurs at any date $t \in [0, T_j]$, and profits/losses from trading (including the initial investment) are expressed in units of the level process $\widetilde{G}(j)$.

4.20 Consistency of Lognormal Market Models

It is essential to observe that Jamshidian's model of swap rates and the lognormal model of forward LIBORs are incompatible with each other.

Indeed, it is not difficult to check that the forward LIBOR and swap rates satisfy

$$\widetilde{\kappa}(t, T_j) = \frac{\prod_{i=j}^{n}(1 + \delta_i L(t, T_i)) - 1}{\sum_{i=j+1}^{n+1} \delta_i \prod_{k=i+1}^{n}(1 + \delta_k L(t, T_k))}.$$

The formula above shows that LIBORs and swap rates cannot have simultaneously deterministic volatilities.

We conclude that the market models for LIBORs and swap rates are inconsistent with each other.

4.21 Market Models: Important Issues

- Calibration of market models of LIBORs and swap rates to implied volatilities and historical correlations.
- Valuation of swaptions in market models of LIBORs.
- Valuation of caps in market models of swap rates.
- Valuation and hedging of exotic products, for instance, barrier caps or Bermudan swaptions.
- Multicurrency extensions of market models.
- Market models with stochastic volatilities and jumps.
- General theory of market models for arbitrary families of market rates.
- Market models with default risk.

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