# An Analytical Pricing Formula for VIX Futures and Its Empirical Applications

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# Outline

- **Background**
  - The introduction to VIX and VIX futures
  - Literature review
  - The motivation and outcomes
- **Analytically pricing VIX futures**
  - Definitions of VIX and VIX futures
  - The Heston stochastic volatility and random jumps model
  - Pricing VIX futures and discussions
- **Empirical investigation**
  - The determination of model parameters.
  - The test of the pricing performance.
- **Concluding remarks**

- **Background**

  The introduction to VIX and VIX futures

  Literature review

  The motivation and outcomes

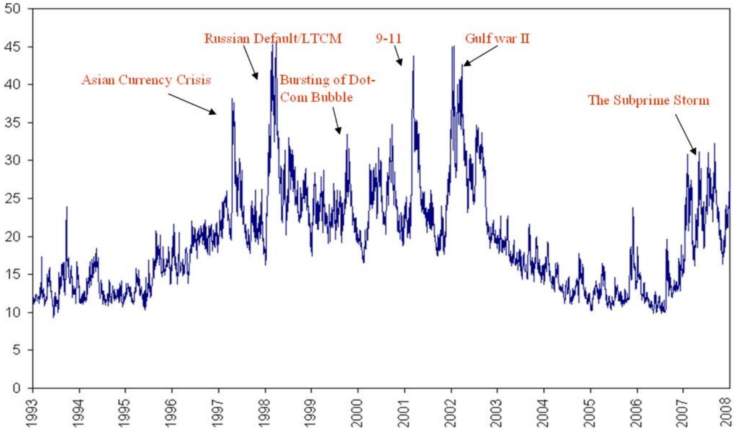
# Background - The story of VIX

- **Volatility Index (VIX) in CBOE**

  "The CBOE Volatility Index (VIX) is a key measure of market expectations of near-term volatility conveyed by $S\&P$500 stock index option prices. Since its introduction in 1993, VIX has been considered by many to be the world's premier barometer of investor sentiment and market volatility."

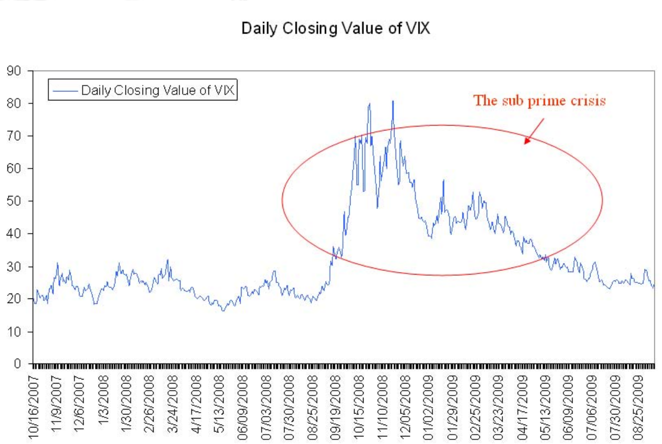
  – Website of CBOE

# Background - The story of VIX




- VIX reflects the market's expectation of the 30-day forward $S\&P500$ index volatility and serves as a proxy for investor sentiment.

# Background - The story of VIX




- VIX rises when investors are anxious or uncertain about the market and falls during times of confidence.

# Background - The story of VIX

- 1993 The VIX Index was introduced in a paper by Professor Robert E. Whaley in Duke University.
- 2003 The VIX methodology was revised.
- 2004 On March 26, 2004, the first-ever trading in futures on the VIX Index began on the CBOE Futures Exchange (CFE).
- 2006 VIX options were launched in February 2006.
- 2008 Binary options on VIX began trading.
- 2009 Mini-VIX futures were launched.

# Background - Trading VIX Futures

- Being warmly welcome by the financial market, VIX futures and VIX options were awarded the most innovative index derivative products.
- "few proposed types of derivatives securities have attached as much attention and interest as futures and options contracts on volatility" (Grunbichler and Longstaff (1996)).
- Investable VIX products could have been used to provide some much needed diversification during the crisis of 2008. VIX calls could have provided a more efficient means of diversification than provided by SPX puts (Szado (2009)).

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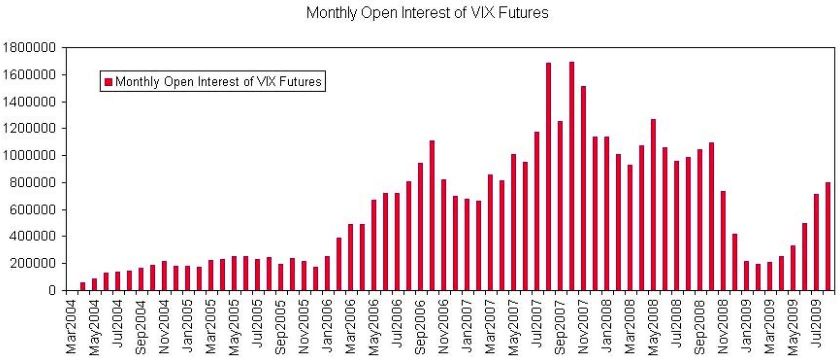
- The open interests of VIX futures in CBOE

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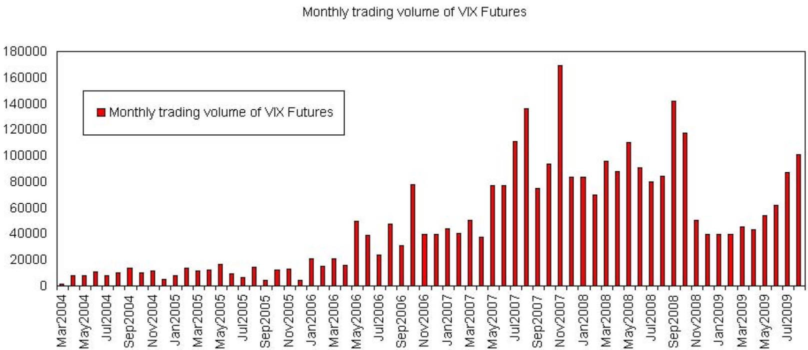
# Background - Trading VIX Futures

- The trading volume of VIX futures in CBOE

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## The financial models

- **Heston (1993):**
  Stochastic volatility without jumps (SV model).
- **Bakshi et al. (1997):**
  Stochastic volatility with jumps in asset return (SVJ model)
  Useful in pricing short-term options, calling for further extension.
- **Sepp (2007):**
  Stochastic volatility with jumps in variance process (SVVJ model)
- **Duffie et al. (2000), Eraker (2004):**
  Stochastic volatility with jumps in asset return and variance process (SVJJ model).
  Received considerable attention, e.g., SVJJ explains the volatility smile for short maturity (Pan 2002).

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**The research interest in pricing VIX futures**

- **Carr and Wu (2006):**
  A lower bound and an upper bound
- **Zhang and Zhu (2006):**
  Exact pricing formula for VIX futures under the Heston (1993) SV model.
  Without paying attention to the jumps.
- **Lin (2007):**
  A convexity adjustment approximation under the SVJJ model.
  As can be shown later, this convexity approximation performs poorly.

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## The motivation of this study

- The quick growth of trading VIX futures in financial market.
- The existing limitations of academic research in pricing VIX futures.

| Models | Research in literature | Results |
|---|---|---|
| SV | Zhang and Zhu (2006) | Exact formula |
| SV | Brenner et al. (2007) | Approximation (third order) |
| SVJJ | Lin (2007) | Approximation (second order) |
| SVJJ | ——— | No exact formula available |

## The outcomes of this research

- Exact pricing formula to price VIX futures in the general SVJJ model, which cover the SV, SVJ, SVVJ as special cases.

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| SVJJ | Found by our research | Exact formula |

- Analyzing the accuracy of the approximation (Lin 2007, Brenner et al. 2007).
- Determining the model parameters with MCMC approach.
- Investigating the pricing performance of these four models (i.e., SV, SVJ, SVVJ, SVJJ).

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  Definitions of VIX and VIX futures

  The Heston stochastic volatility and random jumps model

  Pricing VIX futures and discussions

# Analytically Pricing VIX Futures

We will present a closed-form formula for the pricing of VIX futures, in the framework of SVJJ. Some numerical examples will also be shown to demonstrate the correctness of our exact formula, to investigate the accuracy of the approximation formula in Lin (2007).

- Volatility Index
- Modelling the $S\&P500$ and VIX
- Pricing formula and discussions

# Analytically Pricing VIX Futures - Volatility Index

VIX, which is the underlying of VIX futures and options, is defined by means of $\text{VIX}_t^2$,

$$\text{VIX}_t^2 = \left( \frac{2}{\overline{\tau}} \sum_i \frac{\Delta K_i}{K_i^2} e^{r\overline{\tau}} Q(K_i) - \frac{1}{\overline{\tau}} [\frac{F}{K_0} - 1]^2 \right) \times 100^2$$

- $\overline{\tau} = \frac{30}{365}$,
- $K_i$ is the strike price of the i-th out-of-the-money option written on the S&P500 in the calculation,
- $F$ is the time-t forward S&P500 index level,
- $Q(K_i)$ denotes the time-t midquote price of the out-of-the-money option at strike $K_i$, $K_0$ is the first strike below the forward index level,
- $r$ denotes the time-t risk-free rate with maturity $\overline{\tau}$.

# Analytically Pricing VIX Futures - Volatility Index

For a better understanding of the financial interpretation, this expression of the VIX squared can be presented in terms of the risk-neutral expectation of the log contract,

$$\mathrm{VIX}_t^2 = -\frac{2}{\overline{\tau}} E^{\mathbb{Q}}[\ln\left(\frac{S_{t+\overline{\tau}}}{F}\right)|F_t] \times 100^2 \tag{1}$$

- $\mathbb{Q}$ is the risk-neutral probability measure,
- $F = S_t e^{r\overline{\tau}}$ denotes the 30-day forward price of the underlying S&P500 with a risk-free interest rate $r$ under the risk-neutral probability,
- $F_t$ is the filtration up to time $t$.

# Analytically Pricing VIX Futures - Volatility Index

Volatility Index has an even more intuitive explanation.

- If the $S\&P500$ index does not jump, the VIX squared is the conditional risk-neutral expectation of the annualized realized variance of the S&P500 return over the next 30 calendar days, which means VIX squared can be viewed as an approximation of strike price of the one-month variance swap.

$$\text{VIX}_t^2 = E^{\mathbb{Q}}\left[\lim_{N\to\infty}\frac{1}{\tau}\sum_{i=1}^{N}\log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]\times 100^2 \tag{2}$$

- When jumps are taken into consideration, the VIX squared differs from the one-month realized variance of underlying $S\&P500$.

Under the physical probability measure $\mathbb{P}$, the S&P500 index, is assumed to follow

$$
\begin{cases}
dS_t = S_t(r_t + \gamma_t)dt + S_t\sqrt{V_t}dW_t^S + d(\sum_{n=1}^{N_t} S_{\tau_{n-}}[e^{Z_n^S} - 1]) - S_t\overline{\mu}\lambda dt \\
dV_t = \kappa(\theta - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V + d(\sum_{n=1}^{N_t} Z_n^V)
\end{cases}
$$

- The two standard Brownian motions are correlated with $E[dW_t^S, dW_t^V] = \rho dt$;
- $\kappa$, $\theta$ and $\sigma_V$ are respectively the mean-reverting speed parameter, long-term mean, and variance coefficient of the diffusion $V_t$;
- $N_t$ is the independent Poisson process with intensity $\lambda$, i.e., $Pr\{N_{t+dt} - N_t = 1\} = \lambda dt$, $Pr\{N_{t+dt} - N_t = 0\} = 1 - \lambda dt$. The jumps happen simultaneously in underlying dynamics $S_t$ and variance process $V_t$.

# Analytically Pricing VIX Futures - The Model

Under the physical probability measure $\mathbb{P}$, the S&P500 index, is assumed to follow

$$\begin{cases} dS_t = S_t(r_t + \gamma_t)dt + S_t\sqrt{V_t}dW_t^S + d(\sum_{n=1}^{N_t} S_{\tau_n -}[e^{Z_n^S} - 1]) - S_t\overline{\mu}\lambda dt \\ dV_t = \kappa(\theta - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V + d(\sum_{n=1}^{N_t} Z_n^V) \end{cases}$$

- The jump sizes are assumed to be $Z_n^V \sim \exp(\mu_V)$, and $Z_n^S | Z_n^V \sim N(\mu_S + \rho_J Z_n^V, \sigma_S^2)$;
- $\overline{\mu} = \frac{\exp(\mu_S + \frac{1}{2}\sigma_S^2)}{1 - \rho_J \mu_V} - 1$ is the risk premium of the jump term in the process to compensate the jump component, and $\gamma_t$ is the total equity premium.
- $r_t$ is the constant spot interest rate;
- $V$ is the diffusion component of the variance of the underlying asset dynamics (conditional on no jumps occurring);

# Analytically Pricing VIX Futures - The Model

Under the risk-neutral probability measure $\mathbb{Q}$, the dynamics processes of the S&P500 index and its variance

$$\begin{cases} dS_t = S_t r_t dt + S_t\sqrt{V_t}dW_t^S(\mathbb{Q}) + d(\sum_{n=1}^{N_t(\mathbb{Q})} S_{\tau_{n-}}[e^{Z_n^S(\mathbb{Q})} - 1]) - S_t\overline{\mu}^{\mathbb{Q}}\lambda dt \\ dV_t = \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V(\mathbb{Q}) + d(\sum_{n=1}^{N_t(\mathbb{Q})} Z_n^V(\mathbb{Q})) \end{cases}$$

- $\overline{\mu}^{\mathbb{Q}} = e^{\mu_S^{\mathbb{Q}} + \frac{1}{2}\sigma_S^2}/(1-\rho_J\mu_V)-1$ and $\mu_S^{\mathbb{Q}}$ is the corresponding risk-neutral parameters of $\overline{\mu}$ and $\mu_S$.
- The risk premium parameters are specified as: diffusive volatility risk premium $\eta_V = \kappa^{\mathbb{Q}} - \kappa$ and jump risk premium $\eta_J = \mu_S^{\mathbb{Q}} - \mu_S$.
- the $\sigma_V$, $\rho$, $\kappa\theta$, $\lambda$ and other jumps parameters are the same under both the measure $\mathbb{P}$ and the measure $\mathbb{Q}$.

The four models covered by this general model

$$\begin{cases} dS_t = S_t(r_t + \gamma_t)dt + S_t\sqrt{V_t}dW_t^S + d(\sum_{n=1}^{N_t} S_{\tau_{n-}}[e^{Z_n^S} - 1]) - S_t\overline{\mu}\lambda dt \\ dV_t = \kappa(\theta - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V + d(\sum_{n=1}^{N_t} Z_n^V) \end{cases}$$

- The Heston (1993) SV model: jumps are set to zero, i.e., $\lambda = 0$, $Z_t^S = Z_t^V = 0$.
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VIX squared is the conditional risk-neutral expectation of the log contract of the S&P500 over the next 30 calendar days.

$$\text{VIX}_t^2 = -\frac{2}{\overline{\tau}} E^{\mathbb{Q}}[\ln(\frac{S_{t+\overline{\tau}}}{F})|F_t] \times 100^2 \tag{3}$$

- Under the general specification Eq. (3), this expectation can be carried out explicitly in the form of,

$$\text{VIX}_t^2 = (aV_t + b) \times 100^2 \tag{4}$$

$$\begin{cases} a = \dfrac{1 - e^{-\kappa^{\mathbb{Q}}\overline{\tau}}}{\kappa^{\mathbb{Q}}\overline{\tau}}, \quad and \quad \overline{\tau} = 30/365 \\ b = (\theta^{\mathbb{Q}} + \dfrac{\lambda\mu_V}{\kappa^{\mathbb{Q}}})(1-a) + \lambda c \\ c = 2[\overline{\mu}^{\mathbb{Q}} - (\mu_S^{\mathbb{Q}} + \rho_J\mu_V)] \end{cases}$$

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# Analytically Pricing VIX Futures - Formula

The value of the VIX futures at time $t$ with settlement at time $T$, $F(t,T)$, is calculated by

$$F(t,T) = E^Q[\text{VIX}_T|F_t] = E^Q[\sqrt{aV_T + b}|F_t] \times 100 \tag{5}$$

- Carr and Wu (2006) illustrated that under the assumption of no-arbitrage and continuous marking to market, the VIX futures price, $F(t,T)$, is a martingale under the risk-neutral probability measure $\mathbb{Q}$.
- Lin (2007) and Zhang and Zhu (2006) also concluded that the futures price is a martingale.

# Analytically Pricing VIX Futures - Formula

- We consider the moment generating function, $f(\phi; t, \tau, V_t)$, of the stochastic variable $V_T$, conditional on the filtration $F_t$, with time to expiration $\tau = T - t$.

$$f(\phi; t, \tau, V_t) = E^{\mathbb{Q}}[e^{\phi V_T} | F_t] \tag{6}$$

- Accordingly, the characteristic function is just $f(\phi i; t, \tau, V_t)$.
- Feynman-Kac theorem implies that $f(\phi, \tau)$ must satisfy the following backward partial integro-differential equation (PIDE)

$$\begin{cases} -f_\tau + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V) f_V + \dfrac{1}{2}\sigma^2 V f_{VV} + \lambda E^{\mathbb{Q}}[f(V + Z^V) - f(V) | F_t] = 0 \\ f(\phi; t + \tau, 0, V) = e^{\phi V} \end{cases}$$

# Analytically Pricing VIX Futures - Formula

The above PIDE (7) can be analytically solved and hence we obtain the moment generation function in the form of

$$f(\phi; t, \tau, V_t) = e^{C(\phi,\tau)+D(\phi,\tau)V_t+A(\phi,\tau)} \qquad (7)$$

where

$$\begin{cases} A(\phi,\tau) = \dfrac{2\mu_V\lambda}{2\mu_V\kappa^{\mathbb{Q}} - \sigma_V^2} \ln\left(1 + \dfrac{\phi(\sigma_V^2 - 2\mu_V\kappa^{\mathbb{Q}})}{2\kappa^{\mathbb{Q}}(1-\mu_V\phi)}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ C(\phi,\tau) = \dfrac{-2\kappa\theta}{\sigma_V^2} \ln\left(1 + \dfrac{\sigma_V^2\phi}{2\kappa^{\mathbb{Q}}}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ D(\phi,\tau) = \frac{2\kappa^{\mathbb{Q}}\phi}{\sigma_V^2\phi + (2\kappa^{\mathbb{Q}} - \sigma_V^2\phi)e^{\kappa^{\mathbb{Q}}\tau}} \end{cases}$$

# Analytically Pricing VIX Futures - Formula

The Fourier inversion of the characteristic function $f(\phi i; t, \tau, V_t)$ provides the required conditional density function $p^{\mathbb{Q}}(V_T|V_t)$

$$p^{\mathbb{Q}}(V_T|V_t) = \frac{1}{\pi}\int_0^{\infty} Re[e^{-i\phi V_T} f(i\phi; t, \tau, V_t)]d\phi \quad (8)$$

The price of a VIX future contract at time $t$ is thus expressed in the form of

$$F(t,T) = E^{\mathbb{Q}}[\text{VIX}_T|F_t] = \int_0^{\infty} p^{\mathbb{Q}}(V_T|V_t)\sqrt{aV_T + b}dV_T \times 100 \quad (9)$$

# Analytically Pricing VIX Futures - Formula

This pricing formula can be further simplified by utilizing the expression

$$E[\sqrt{x}] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E[e^{-sx}]}{s^{\frac{3}{2}}} ds \tag{10}$$

Invoking this identity, Formula (9) can be simplified as

$$F(t,T) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-sb} f(-sa; t, \tau, V_t)}{s^{\frac{3}{2}}} ds \tag{11}$$

where $f(\phi; t, \tau, V_t)$ is the moment generating function shown in Eq. (7).

**The closed-form pricing formula for VIX futures**

$$F(t,T) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-sb} f(-sa; t, \tau, V_t)}{s^{\frac{3}{2}}} ds \tag{12}$$

where $f(\phi; t, \tau, V_t)$ is the moment generating function shown in Eq. (7).

- A closed-form solution;
- Efficient and exact;
- Useful in empirical study: model calibration;

**Property of VIX futures**

- The standard convergence property of futures prices to the underlying spot value at maturity

$$\lim_{(T-t)\to 0} F(t,T) = \mathrm{VIX}_t \tag{13}$$

- The quite unique property

$$\lim_{(T-t)\to \infty} F(t,T) = Constant \tag{14}$$

The constant is independent of the underlying VIX and $T-t$.

# Analytically Pricing VIX Futures - Discussions

We now show some numerical results to illustrate the properties of our newly-found VIX futures pricing formula, by comparing the results obtained from

- the implementation our VIX futures pricing formula, Eq. (12).
- the Monte Carlo simulations to verify the correctness of our newly-found formula.
- the results obtained from the convexity correction approximations (e.g., Lin (2007) and Brenner et al. (2007)). These comparisons will help readers understand the improvement in accuracy of our exact solution.

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# Analytically Pricing VIX Futures - Discussions

For simplicity, we have employed the simple Euler-Maruyama discretization for the variance dynamics:

$$v_t = v_{t-1} + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - v_{t-1})\Delta t + \sigma\sqrt{|v_{t-1}|}\sqrt{\Delta t}W_t + \sum_{n=1}^{N_t} Z_n^V \qquad (15)$$

- $W_t$ is a standard normal random variables
- $Z_n^V \sim \exp(\mu_V)$,
- $N_t$ is the independent Poisson process with intensity $\lambda\Delta t$.

# Analytically Pricing VIX Futures - Discussions

By using the convexity correction approximation proposed by Brockhaus and Long (2000), Lin (2007) was able to present the VIX futures approximation formula in the form of

$$F(t,T) = E^{\mathbb{Q}}[\text{VIX}_T|F_t] \approx \sqrt{E_t^{\mathbb{Q}}(\text{VIX}_T^2)} - \frac{var^{\mathbb{Q}}(\text{VIX}_T^2)}{8[E^{\mathbb{Q}}(\text{VIX}_T^2)]^{\frac{3}{2}}} \tag{16}$$

- $var^{\mathbb{Q}}(\text{VIX}_T^2)/\{8[E^{\mathbb{Q}}(\text{VIX}_T^2)]^{\frac{3}{2}}\}$ is the convexity adjustment relevant to the VIX futures.
- This formula is indeed very easy to be implemented.
- But what about the accuracy?

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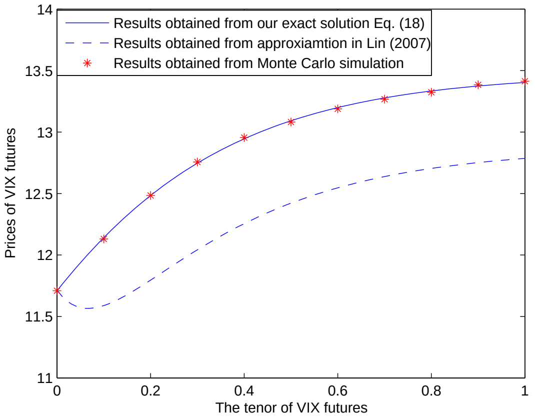
# Analytically Pricing VIX Futures - Discussions

- Numerical Comparison of VIX futures prices



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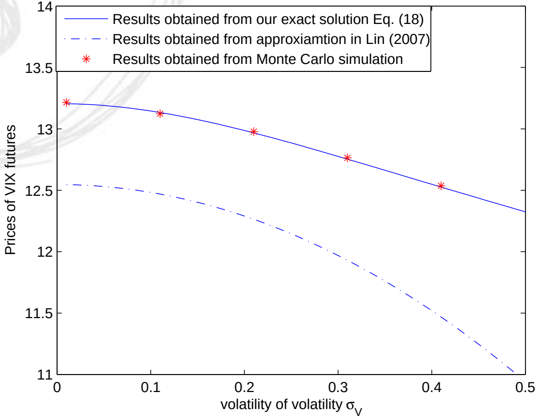

# Analytically Pricing VIX Futures - Discussions

- Lin (2007)'s convexity correction approximation is essentially a Taylor-series expansion of the square root function to the second order.
- Brenner et al. (2007) explored a third order Taylor expansion of the square root function and obtained an approximation formula for VIX futures, based on the Heston SV model.
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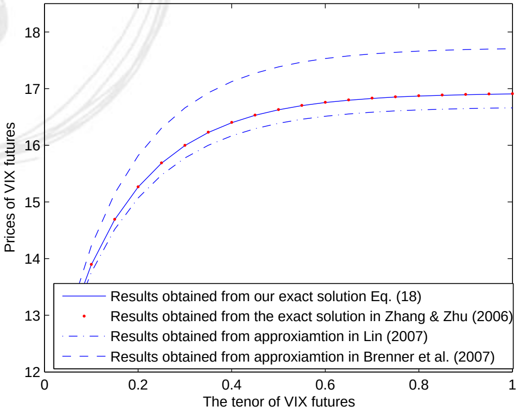


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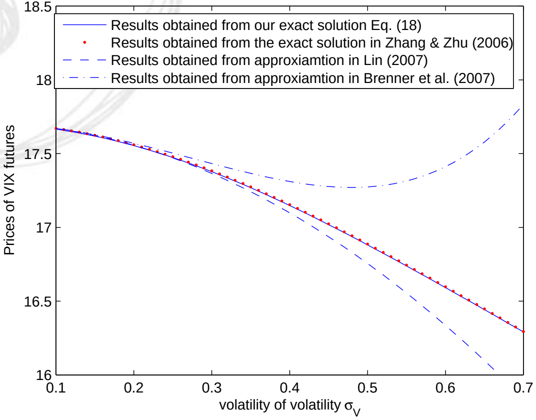

# Empirical Investigation

- **Background**

  The introduction to VIX and VIX futures

  Literature review

  The motivation and outcomes

- **Analytically pricing VIX futures**

  Definitions of VIX and VIX futures

  The Heston stochastic volatility and random jumps model

  Pricing VIX futures and discussions

- **Empirical investigation**

  The determination of model parameters.

  The test of the pricing performance.

# Empirical Investigation

- Like any other pricing formulae, to apply our formula to price VIX futures in practice, one needs to know what parameters to use.
- The determination of the model-needed parameters in a proper and sensible way can itself be a difficult problem.
- Furthermore, one now naturally has to choose the most suitable one to price VIX futures, among the four available models (SV, SVJ, SVVJ and SVJJ models).
- The working out our VIX pricing formula allows us to empirically obtain model parameters and compare which model is the most suitable one to price VIX futures(SV, SVJ, SVVJ and SVJJ).

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We choose the Markov Chain Monte Carlo (MCMC) method to estimate the model parameters.

- the calibration method (the optimization method) by minimizing the squared differences between theoretical values calculated from any VIX futures model and those observed in the market, appears to be unstable.
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The Markov chain Monte Carlo (MCMC) method to estimate the model parameters

- As commented by Broadie et al. (2007), an efficient estimation procedure should utilize not only the information stored in the underlying that varies as a function of time over the period of study but also the cross-sectional information stored in the derivatives prices over the same period of time.
- Using the joint data of underlying and the cross-sectional derivatives prices were used to estimate the model parameters, the MCMC method naturally became our selected method to conduct the empirical study.

# Empirical Investigation - Econometric Methodology

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# Empirical Investigation - Econometric Methodology

MCMC method is a Bayesian inference, which utilizes the observable data to estimate the distribution of parameters and unobservable variables, $p(\Theta, X|Y, VIX)$.

- Normally, researches have a prior perspective.
- Construct the conditional probability density function, and draw a sample from it.

$$\begin{array}{ll} 1.Draw & \Theta^{(1)}(\Theta|X^{(0)}, Y, VIX) \\ 2.Draw & X^{(1)}(\Theta|\Theta^{(1)}, Y, VIX) \end{array} \quad (17)$$

- This algorithm is running until it has converged (Markov Chain).
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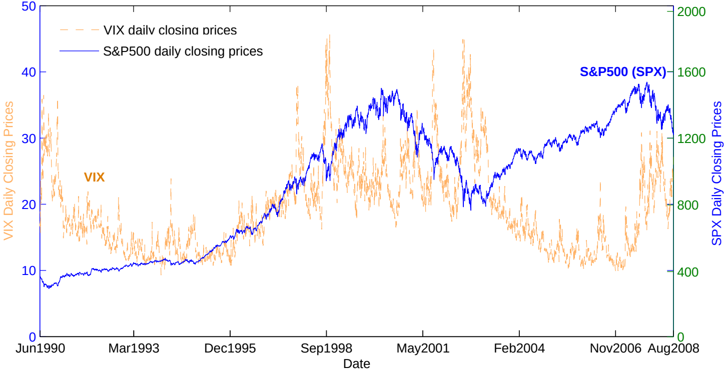
To use the MCMC method to estimate the structural parameters, we construct a time-discretization of Eq. (3).

$$\begin{cases} Y_t = \mu + \sqrt{V_{t-1}}\varepsilon_t^S + Z_t^S dq \\ V_t = V_{t-1} + \kappa(\theta - V_{t-1}) + \sigma_V \sqrt{V_{t-1}}\varepsilon_t^V + Z_t^V dq \\ \text{VIX}_t^2 = (aV_t + b) \times 100^2 + \varepsilon_t^{VIX} \end{cases} \tag{18}$$

- $dq$ =1 indicates a jump arrival,
- $\varepsilon_t^S$ and $\varepsilon_t^V$ are standard normal random variables with correlation $\rho$,
- $Y_t$ are continuous daily returns, e.g., $Y_t = \ln(S_t/S_{t-1})$.
- All the parameters are quoted using a daily time interval following the convention in the time-series literature.
- This algorithm is implemented using WinBUGS.

# Empirical Investigation - Data

- Joint data of S&P500 and VIX used to obtain model parameters.

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# Empirical Investigation

- The obtained model parameters using the MCMC

Table: Parameters Estimates

| Parameters | SV | SVJ | SVVJ | SVJJ |
|---|---|---|---|---|
| $\theta$ | 1.761 | 1.684 | 1.624 | 1.541 |
| $\kappa^{\mathbb{Q}}$ | 0.009 | 0.009 | 0.007 | 0.008 |
| $\sigma_V$ | 0.153 | 0.120 | 0.136 | 0.045 |
| $\eta_V$ | -0.008 | -0.010 | -0.007 | -0.007 |
| $\rho$ | -0.753 | -0.668 | -0.766 | -0.577 |
| $\lambda$ | | 0.002 | 0.001 | 0.0007 |
| $\mu_S^{\mathbb{Q}}$ | | -0.510 | | -0.736 |
| $\sigma_S$ | | 2.007 | | 2.305 |
| $\mu_V$ | | | 2.044 | 0.374 |
| $\eta_J$ | | -0.101 | | -0.218 |
| $\rho_J$ | | | | 0.422 |

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# Empirical Investigation

To assess the pricing performance, we employ three measures of "goodness of fitting", by comparing model-determined future price with the observed market counterpart

- root mean squared error (RMSE)
- mean absolute error (MAE)
- mean percentage error (MPE)

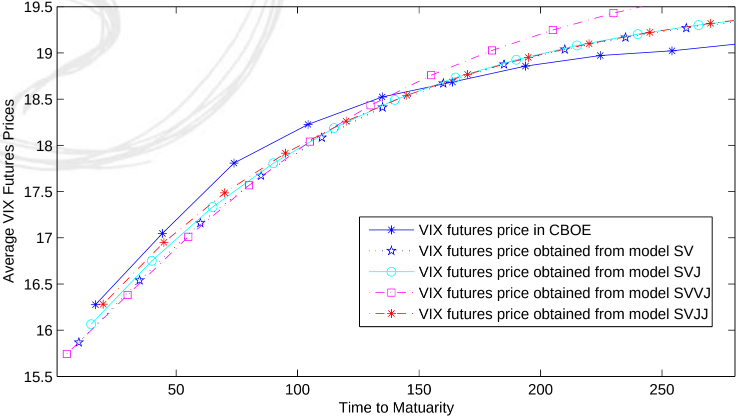
# Empirical Investigation

- The Pricing Performance of the Four Models

| | | Time to Expiration | | | |
|---|---|---|---|---|---|
| Pricing Errors | Models | All Futures | <=60 | 60-180 | >=180 |
| RMSE | SV | 2.668 | 1.782 | 2.940 | 3.230 |
| | SVJ | 2.615 | 1.731 | 2.856 | 3.198 |
| | SVVJ | 2.578 | 1.633 | 2.718 | 3.271 |
| | SVJJ | 2.485 | 1.625 | 2.657 | 3.095 |
| MPE(%) | SV | 5.399 | 2.880 | 5.112 | 8.651 |
| | SVJ | 5.624 | 3.174 | 5.340 | 8.790 |
| | SVVJ | 6.184 | 2.556 | 5.855 | 10.790 |
| | SVJJ | 5.774 | 3.303 | 5.514 | 8.942 |
| MAE | SV | 2.343 | 1.479 | 2.713 | 3.037 |
| | SVJ | 2.296 | 1.443 | 2.635 | 3.006 |
| | SVVJ | 2.237 | 1.335 | 2.505 | 3.068 |
| | SVJJ | 2.174 | 1.351 | 2.449 | 2.907 |

# Empirical Investigation

- Numerical Comparison of VIX futures prices

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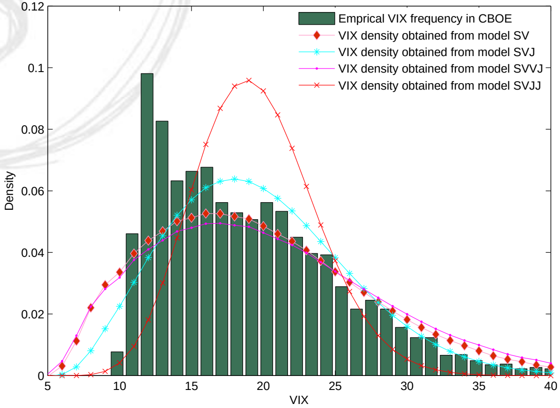

# Concluding Remarks

- A closed-form exact solution is presented for the pricing of VIX futures, based on the SVJJ model. It has filled up a gap that there is no closed-form exact solution available in the SVJJ model.
- We found out that the approximation formulae in the literature perform poorly.
- Using the joint data of S&P500 and VIX, we have demonstrated the determination of model parameters with the MCMC approach.
- We have empirically examined the pricing performance of the four models (SV, SVJ, SVVJ and SVJJ).
- Our empirical studies show that the SV model can well capture the dynamics of S&P500 already and is a good candidate for the pricing of VIX futures. Incorporating jumps into the underlying price can further improve the pricing performance for VIX futures.

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# Reference

Bakshi, G., Cao, C. and Chen, Z. (1997), 'Empirical Performance of Alternative Option Pricing Models', *The Journal of Finance* **52**(5), 2003–2049.

Black, F. and Scholes, M. (1973), 'The pricing of options and corporate liabilities', *J. of Political Economy* **81**, 637–654.

Brenner, M., Shu, J. and Zhang, J. (2007), 'The Market for Volatility Trading; VIX Futures', *working paper, availble at http://archive.nyu.edu/handle/2451/26289* .

Broadie, M., Chernov, M. and Johannes, M. (2007), 'Model Specification and Risk Premia: Evidence from Futures Options', *The Journal of Finance* **62**(3), 1453–1490.

Brockhaus, O. and Long, D. (2000), 'Volatility swaps made simple', *Risk* **19**(1), 92–95.

# Reference


Carr, P. and Wu, L. (2006), 'A Tale of Two Indices ', *Journal of Derivatives* **13**(3), 13–29.

Duffie, D., Pan, J. and Singleton, K. (2000), 'Transform Analysis and Asset Pricing for Affine Jump-diffusions', *Econometrica* **68**(6), 1343–1376.

Eraker, B. (2004), 'Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices', *The Journal of Finance* **59**(3), 1367–1404.

Grunbichler, A. and Longstaff, F. (1996), 'Valuing Futures and Options on Volatility', *Journal of Banking and Finance* **20**(6), 985–1001.

Heston, S. (1993), 'A closed-form solution for options with stochastic volatility with applications to bond and currency options', *Review of Financial Studies* **6**(2), 327–343.

# Reference

Lin, Y. (2007), 'Pricing VIX futures: Evidence from integrated physical and risk-neutral probability measures', *JOURNAL OF FUTURES MARKETS* **27**(12), 1175.

Sepp, A. (2007), 'Affine models in mathematical finance: an analytical approach'.

Szado, E. (2009), 'VIX Futures and Options-A Case Study of Portfolio Diversification During the 2008 Financial Crisis', *Working Paper, available SSRN* .

Zhang, J. and Zhu, Y. (2006), 'VIX Futures', *Journal of Futures Markets* **26**(5), 521–531.

*Thank you!*


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