Options on Renewable Resources: A New Version of the Black (1976) Pricing Formula for Commodity Options

Christian-Oliver Ewald

University of Sydney
School of Mathematical Sciences

March 9th, 2010

Sydney Financial Mathematics Workshops

Outline

Introduction

- With commodities in general, but renewable resources in particular, it is not realistic to assume that the resource is traded at all times
  - agricultural products (seasonality)
  - fish (biological seasonality and regulated seasonality)
- forward and future markets for renewable resources exist, but when modeling these, it does not appear to be appropriate to specify future prices exogenously (as a geometric Brownian motion for example, Black (1976))

We assume that the supply of the renewable resource follows a stochastic logistic growth dynamics

$$dX(t) = \kappa X(t)(\theta - X(t))dt + \sigma X(t)dW(t). \tag{1}$$

with $\kappa\theta > \sigma^2$.

Introduction

We assume that the renewable resource is not tradable, except at one time in the future T, when it is traded for a certain price $P(T)$.

1. we think of this price developing over time $t \in [0, T]$ though, and call $P(t)$ the price of the ressource
2. later we will also look at the forward price of the renewable resource and in addition consider forward contracts and options

We assume that the price of the resource is given by

$$P(t) = \frac{p}{X(t)} \tag{2}$$

where p is a constant (supply and demand feature).

Introduction

Derivatives pricing is done under a no-arbitrage assumption:

1. assume that the dynamics (1) is specified under a chosen risk neutral measure $\mathbb{P}$
2. alternatively assume that there is a full spanning asset $S(t)$ which is tradable and follows a geometric Brownian motion

$$dS(t) = S(t)(rdt + \sigma dW(t)), \tag{3}$$

For the case of forward prices and forward contracts, we derive closed form expressions.

1. the assumption made in Black (1976) of forward prices evolving according to a geometric Brownian motion is highly questionable
2. we will observe, that there is a relation between the price process $P(t)$ and the underlying of an Asian (Australian) option

Introduction

This Asian (Australian) feature enables us to come up with an approximate option pricing formula for a European call on the renewable resource:

1. this formula has the structure of the Black (1976) formula, except that the Normal distribution function is replaced by the reciprocal Γ-distribution function
2. it is essentially very similar to the one derived by Milevsky and Posner (1998) for Asian options

We also derive an integral representation for the true price of the option using results obtained in Yang and Ewald (2010).Further

1. we derive an approximate option pricing formulas for a European call on a forward contract (after initiation)
2. we discuss how the price of the renewable resource $P(t)$ as well as the forward contract can be hedged under the assumption that there is a spanning asset (3).

An Australian Transformation and the Moments of $P(t)$

The payoff of an Australian option is given by

$$H = h\left(\frac{\int_0^T S(t)dt}{S(T)}\right). \tag{4}$$

1. these options are traded in Australia and have been studied in Handley (2000), Handley (2003) and Moreno Navas (2008)
2. a computation shows that

$$P(t) = p\frac{1 + \kappa\int_0^t \tilde{S}(s)ds}{\tilde{S}(t)}. \tag{5}$$

with

$$d\tilde{S}(t) = \tilde{S}(t)((r-\rho)dt + \sigma dW(t)) \tag{6}$$

and $\rho = r - \kappa\theta$.

An Australian Transformation and the Moments of $P(t)$

Lemma (Moreno and Navas (1998))

With $\tilde{S}(t)$ given by equation (6) it holds that

$$\mathbb{E}\left(\frac{\int_0^T \tilde{S}(u)du}{\tilde{S}(T)}\right) = \phi\left(\sigma^2 - \kappa\theta\right)$$

$$\mathbb{E}\left(\frac{\int_0^T \tilde{S}(u)du}{\tilde{S}(T)^2}\right) = \frac{\phi(\kappa\theta - 2\sigma^2)}{\tilde{S}(0)e^{(2\kappa\theta - 3\sigma^2)T}}$$

$$\mathbb{E}\left(\frac{\left(\int_0^T \tilde{S}(u)du\right)^2}{\tilde{S}(T)^2}\right) = \frac{\phi(2\kappa\theta - 3\sigma^2) - \phi(\kappa\theta - 2\sigma^2)}{2e^{(2\kappa\theta - 3\sigma^2)T}\left(\kappa\theta - \sigma^2\right)}$$

with $\phi(x) = \frac{e^{xT}-1}{x}$ for $x \neq 0$ and $\phi(0) = T$.

An Australian Transformation and the Moments of $P(t)$

From the Lemma we conclude:

$$\mathbb{E}(P(T)) = P(0)e^{-(\kappa\theta-\sigma^2)T} + \frac{p\kappa}{\kappa\theta-\sigma^2}\left(1 - e^{-(\kappa\theta-\sigma^2)T}\right) \quad (7)$$

$$\mathbb{E}(P(T)^2) = pe^{-2(\kappa\theta-3\sigma^2)T}\left[\frac{1}{\tilde{S}(0)^2} + 2p\kappa\frac{1}{\tilde{S}(0)}\phi(\kappa\theta-2\sigma^2) + 2p\kappa^2\frac{\phi(2\kappa\theta-3\sigma^2)-\phi(\kappa\theta-2\sigma^2)}{\kappa\theta-\sigma^2}\right] \quad (8)$$

1. the result on the first moment will enable us to infer on the forward price of the renewable ressource
2. both the first and second moment will be used to calibrate our approximate pricing formula in the spirit of Milevsky and Posner (1998) (i.e. moment matching)

Forward Prices and the Value of Forward Contracts

Forward contract:

1. agreement established at a time $s < T$ to deliver or receive the renewable ressource for a price K, which is specified at time s
2. in financial terms, the payoff at time T of such a forward contract is
$$H = P(T) - K. \tag{9}$$
3. the arbitrage free value of this payoff and hence the value of the contract at any time $s \leq t \leq T$ is given by
$$V_F(t) = e^{-r(T-t)}\mathbb{E}\left(P(T) - K|\,\mathcal{F}_t\right). \tag{10}$$
4. the forward price $F_P(s,T)$ at time s is the price K agreed at time s which makes the value of the forward contract zero at time s, in our case
$$F_P(s,T) = \mathbb{E}\left(P(T)|\mathcal{F}_s\right) \tag{11}$$

Forward Prices and the Value of Forward Contracts

1. note that the forward contract with forward price $F_P(s,T)$ that has been agreed on at time s, is only worthless at time s, after that it can take any value, positive or negative
2. also note, that forward prices are not the prices of any tradable assets
3. but: forward contracts are traded after their initiation, which means that the discounted value process of any forward contract $V_F(t)$ needs to be a martingale on $[s,T]$ (see (10))

Proposition

The forward price at time s of the renewable ressource for delivery at time T is given by

$$F_P(s,T) = P(s)e^{-(\kappa\theta-\sigma^2)(T-s)} + \frac{p\kappa}{\kappa\theta-\sigma^2}\left(1-e^{-(\kappa\theta-\sigma^2)(T-s)}\right). \tag{12}$$

Forward Prices and the Value of Forward Contracts

Black (1976) postulates that forward prices should evolve according to a geometric Brownian motion with zero drift term:

1. based on this assumption he derives a version of the Black-Scholes formula for commodity prices
2. it has been discussed, whether given the fact that geometric Brownian motion is certainly not an adequate model for commodity prices, it can be used as a realistic model for forward prices of commodities

We obtain:

$$dF_p(s,T) = F_P(s,T)\sigma dW(s) + \frac{p\kappa}{\kappa\theta - \sigma^2}\left(e^{-(\kappa\theta-\sigma^2)(T-s)} - 1\right) dW(s). \quad (13)$$

While this is still a linear SDE and its solution can be computed relatively easy, it is not a geometric Brownian motion.

Forward Prices and the Value of Forward Contracts

Using the relationship between the value of the forward and the forward price:

$$
\begin{aligned}
V_F(t) &= e^{-r(T-t)} \cdot \mathbb{E}\left(P(T) - F_P(s,T) \middle| \mathcal{F}_t \right) \\
&= e^{-r(T-t)} \cdot \left(\mathbb{E}\left(P(T) \middle| \mathcal{F}_t \right) - F_P(s,T) \right) \\
&= e^{-r(T-t)} \cdot \left(F_P(t,T) - F_P(s,T) \right).
\end{aligned}
$$

we obtain for the dynamic of the value of the forward contract

$$
\begin{aligned}
dV_F(t) &= d\left(e^{-r(T-t)} F_p(t,T) \right) - d\left(e^{-r(T-t)} F_p(s,T) \right) \\
&= re^{-r(T-t)} \left(F_P(t,T) - F_P(s,T) \right) dt + e^{-r(T-t)} dF_P(t,T) \\
&= re^{-r(T-t)} V_F(t) dt + e^{-r(T-t)} F_P(t,T) \sigma dW(t) \\
&\quad + e^{-r(T-t)} \frac{p\kappa}{\kappa\theta - \sigma^2} \left(e^{-(\kappa\theta - \sigma^2)(T-t)} - 1 \right) dW(t).
\end{aligned}
$$

with $V_F(s) = 0$.

An Approximative Option Pricing Formula

We consider the problem of pricing a European call option with payoff

$$H = (P(T) - K)^{+} \tag{14}$$

as well as European call options, which are written on the value of a forward contract.

1. deriving a pricing PDE following the same approach as Black (1976) and taking the dynamic (13) into account is possible, but leads to a PDE, which is far more complex than the Black-Scholes equation and can not be solved in closed form
2. at this point we do not attempt to compute a closed form expression for the exact option price, but instead derive an approximation, which as we demonstrate later, produces excellent results
3. this approach is motivated by Milevsky and Posner's idea, to use the reciprocal Γ- distribution to approximate Asian option prices, see Milevsky and Posner (1998)

An Approximative Option Pricing Formula

1. under the condition $\kappa\theta > \sigma^2$ the geometric mean reversion process $X(t)$ from equation (1) admits an equilibrium distribution (intuitively the distribution of $X(\infty)$)
2. this distribution is the Γ-distribution, i.e.

$$X(\infty) \sim \Gamma(k, \delta)$$

   with $k = \frac{2\kappa\theta}{\sigma^2} - 1$, $\delta = \frac{\sigma^2}{2\kappa}$
3. The density $\pi(k, \delta)(x)$ of $\Gamma(k, \delta)$ is given by

$$\pi(k, \delta)(x) = \begin{cases} x^{k-1} \frac{e^{-x/\delta}}{\delta^k \Gamma(k)} & x > 0 \\ 0 & x \leq 0. \end{cases} \tag{15}$$

An Approximative Option Pricing Formula

Assume that time to maturity T of an option contract $h(P(T))$ as in (14) is large, then it makes sense to assume that at expiry the distribution of $X(T)$ is very close to its equilibrium distribution and that we have

$$\mathbb{E}\left(e^{-rT}h(P(T)\right) \approx \mathbb{E}\left(e^{-rT}h(P(\infty)\right), \tag{16}$$

where according to equation (2) we have $P(\infty) = \frac{p}{X(\infty)}$.

1. Example: In the case of a fish population, this means that the population is practically assumed to be in equilibrium at the start of the catch season, which in many fish species is a rather realistic assumption.

The usefulness of equation (16) depends on how difficult it is to compute the right hand side: In standard cases such as a European call or put on $P(T)$, this can be easily done.

An Approximative Option Pricing Formula

Proposition

The option pricing formula for a European call on the price of a renewable ressource $P(T)$ obtained by replacing the distribution of $X(T)$ with its equilibrium distribution is given by

$$\Pi_P = F \cdot \Gamma\left(\frac{2\kappa\theta}{\sigma^2} - 2, \frac{\sigma^2}{2\kappa}, \frac{p}{K}\right) - e^{-rT} K \cdot \Gamma\left(\frac{2\kappa\theta}{\sigma^2} - 1, \frac{\sigma^2}{2\kappa}, \frac{p}{K}\right) \tag{17}$$

with $F = e^{-rT}\frac{p\cdot\kappa}{\kappa\theta-\sigma^2} = e^{-rT}\mathbb{E}(P(\infty))$ being the discounted expectation of the equilibrium distribution.

1. analogy to Black-Scholes: the normal distribution is replaced by the Γ-distribution, and the spot price by the discounted expectation of the equilibrium distribution
2. note that the option pricing formula (17) does not depend on the initial price of the underlying (not a surprise)

An Approximative Option Pricing Formula

We compared (17) with a high accuracy Monte-Carlo simulation

case	relative error
$\kappa = 1$, $T = 5$	-0.69% to -0.5%
$\kappa = 1$, $T = 1$	-1.51% to -1.15%
$\kappa = 1$, $T = 0.05$	around 2%
$\kappa = 0.2$, $T = 5$	around 3%
$\kappa = 0.2$, $T = 0.05$	around 17%

The table and figure on the following page show, that our approximation is extremely accurate, except in the case where mean reversion speed is very low, and time to maturity very short.

An Approximative Option Pricing Formula

A simple but effective mean to improve on this, is to choose the parameters κ and θ used in the Γ distribution to reflect mean and variance of the actual underlying $P(T)$ rather than $P(\infty)$:

1. we have computed the first two moments of $P(T)$ (equation (8))
2. the first two moments M_1 and M_2 of the reciprocal Γ-distribution with parameters δ and k are given by

$$M_1 = \frac{1}{\delta(k-1)} \tag{18}$$

$$M_2 = \frac{1}{\delta^2(k-1)(k-2)}. \tag{19}$$

3. we then chose k and δ such that M_1 and M_2 match the values in (8) and use equation (17) with the obtained values for k and δ to price the option

An Approximative Option Pricing Formula

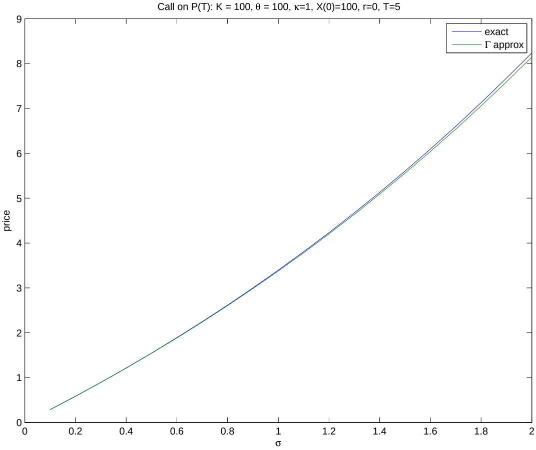

Figure: Call on price of renewable ressource $P(T)$ via Γ-approximation

An Approximative Option Pricing Formula

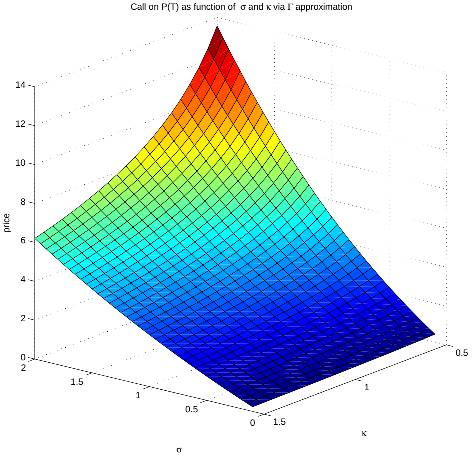

Figure: Call on price of renewable ressource $P(T)$ via Γ-approximation

An Analytic Option Pricing Formula

Proposition (Yang and Ewald (2010))

Let $X(t)$ be as defined by (1). Its density function $\tilde{p}_t(x)$ is given by

$$\tilde{p}_t(x) = \frac{\sigma^2}{4\kappa x^2} \exp\left(-\frac{(\frac{\sigma^2}{2} - \kappa\theta)^2 t}{2\sigma^2}\right) \times$$

$$\int_{-\infty}^{\infty} \exp\left(2\left(1 - \frac{\frac{\sigma^2}{2} - \kappa\theta}{\sigma^2}\right) z\right) f_{\sigma^2 t/4}\left(\frac{\sigma^2 \exp(2z) - 4xY(0)}{4\kappa x}, z\right) dz$$

for $y > 0$, $\tilde{p}_t(y) = 0$ *for* $y \leq 0$. *Here* $f_t(x, y) = 0$ *for* $x \leq 0$, *and*

$$f_t(x,y) = \rho_t(x,y) \int_0^{\infty} \exp\left(-\frac{z^2}{2t} - \frac{\exp(y)}{x}\cosh(z)\right) \sinh(z) \sin\left(\frac{\pi z}{t}\right) dz \tag{20}$$

for $x > 0$, $\rho_t(x,y) = \left(x^2\sqrt{2\pi^3 t}\right)^{-1} \exp\left(\frac{2xyt + \pi^2 x - t - t\exp(2y)}{2xt}\right)$.

An Analytic Option Pricing Formula

Corollary

Let $P(t)$ be the price of the renewable ressource defined as in (2), then it's density is given by

$$p_t(p) = \frac{\sigma^2}{4\kappa} \exp\left(-\frac{(\frac{\sigma^2}{2} - \kappa\theta)^2 t}{2\sigma^2}\right) \times$$

$$\int_{-\infty}^{\infty} \exp\left(2\left(1 - \frac{\frac{\sigma^2}{2} - \kappa\theta}{\sigma^2}\right) z\right) f_{\sigma^2 t/4}\left(\frac{p\sigma^2 \exp(2z) - 4X(0)}{4\kappa}, z\right) dz$$

for $y > 0$ and $p_t(p) = 0$ for $y \leq 0$.

Given that we know the density function of the underlying, we can in principle compute the price of any derivative on this underlying by numerical integration.

Options on Forward Contracts

We look at European calls with expiry $s < T_1 < T$ on the value of a forward contract entered at time s with delivery at time T:

$$H = (V_F(T_1) - K)^+ . \tag{21}$$

Why would a trader in commodities be interested in such a product:

1. the value of a forward contract at initiation is zero, but after initiation it changes, and can be positive or negative at any time $t > s$
2. a forward contract is not like an option, it represents an obligation for both parties
3. the downside risk for both parties (long/short) is unlimited
4. to control the downside risk, the party who enters the short position for example, may buy the option (21), and at time T_1 have a position which is worth

$$(V_F(T_1) - K)^+ - V_F(T_1) \geq -K. \tag{22}$$

Options on Forward Contracts

1. buying a whole portfolio of these options with expiries $s < T_1 < T_2 < \dots < T_n < T$ a commodity trader is able to curb his downside risk almost completely

Proposition

The price of a European call option with strike K traded at time s with expiry $s < T_1 < T$ written on a forward contract entered at time s under the forward price $F_P(s, T)$ is approx. given by

$$\Pi_P = \tilde{F} \cdot \Gamma\left(\frac{2\kappa\theta}{\sigma^2} - 2, \frac{\sigma^2}{2\kappa}, \frac{p}{\tilde{K}}\right) - e^{-(rT_1 + \alpha(T - T_1))} \tilde{K} \cdot \Gamma\left(\frac{2\kappa\theta}{\sigma^2} - 1, \frac{\sigma^2}{2\kappa}, \frac{p}{\tilde{K}}\right)$$

with
$F = e^{-(r(T_1 - s) + \alpha(T - T_1))} \frac{p \cdot \kappa}{\kappa\theta - \sigma^2} = e^{-(r(T_1 - s) + \alpha(T - T_1))} \mathbb{E}(P(\infty))$
being the appropriately discounted expectation of the equilibrium distribution.

Hedging the Renewable Ressource and Forward Contracts

1. we assume the existence of a full spanning asset $S(t)$ following the dynamic (3) and show how it is possible to replicate the price $P(t)$ of the non-traded renewable ressource, by trading in the spanning asset and a bank account $B(t)$
2. this strategy then represents a perfect hedge
3. to derive this hedge we use basic Malliavin calculus (in particular the Clark-Ocone formula)

After a little computation we conclude from the Clark-Ocone formula:

$$\begin{aligned} P(T) &= \mathbb{E}(P(T)) + \int_0^T \mathbb{E}(D_t P(T)|\mathcal{F}_t) dW_t \qquad (23) \\ &= \mathbb{E}(P(T)) - \int_0^T p\sigma \frac{\left(1 + \kappa \int_0^t \tilde{S}(u) du\right)}{\tilde{S}(t)} e^{\left(\left(\sigma^2 - \kappa\theta\right)(T-t)\right)} dW_t. \end{aligned}$$

Hedging the Renewable Ressource and Forward Contracts

Proposition

The dynamic hedging strategy for $P(T)$ satisfying

$$P(T) = P(0) + \int_0^T \phi^0(t)dB(t) + \int_0^T \phi^1(t)dS(t)$$

is given by

$$\begin{aligned}
\phi^0(t) &= \frac{1}{r}e^{-rt}\left(\frac{1}{\kappa\theta-\sigma^2}\left(\frac{p\kappa}{\kappa\theta-\sigma^2} - P(0)\right)e^{-(\kappa\theta-\sigma^2)t}\right) \\
&\quad -p\frac{\left(1+\kappa\int_0^t e^{-\rho u}S(u)du\right)}{e^{\kappa\theta t}S(t)}\exp\left(\left(\sigma^2-\kappa\theta\right)(T-t)\right) \\
\phi^1(t) &= -p\frac{\left(1+\kappa\int_0^t e^{-\rho u}S(u)du\right)}{e^{-\rho t}S(t)^2}\exp\left(\left(\sigma^2-\kappa\theta\right)(T-t)\right).
\end{aligned}$$

Hedging the Renewable Ressource and Forward Contracts

1. the formula above would hold for general T, but note that as T enters explicitly in the hedge, one would obtain different hedges for every T
2. the trading strategy computed above hence is not a tracking strategy for the price process $P(t)$
3. for the portfolio value we have $V_\phi(T) = P(T)$ but the law of one price, from which one could incorrectly conclude $V_\phi(t) = P(t)$, does not apply, as the commodity is not tradable at any time prior to t
4. the correct conclusion comes from the no-arbitrage principle, which implies that

$$V_\phi(t) = e^{-r(T-t)}\mathbb{E}\left(P(T)|\,\mathcal{F}_t\right) = e^{-r(T-t)}F_P(t,T).$$

and hence $V_\phi(t) = V_F(t) + e^{-r(T-t)}F_P(s,T)$.

Hedging the Renewable Ressource and Forward Contracts

From the latter it is easy to construct a hedge for the forward contract:

1. do nothing until time s (the forward contract is only initiated at time s)
2. at time s borrow $e^{-r(T-s)}F_P(s,T)$ from the bank and invest the proceeds according to the strategy $\phi = (\phi^0(t), \phi^1(t))$ from Proposition 7.1.
3. wait and see !

full paper is a vaialbale at:

SSRN: http://ssrn.com/abstract=1469135