Stochastic Portfolio Theory

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Main references:

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The basic framework

- n stocks, caps (or prices) $X_i(t)$.

- $m \geq n$ independent BMs $W_i(t)$.

- Stock prices/caps satisfy

$$dX_i(t) = X_i(t)\left(b_i(t)\,dt + \sum_{\nu=1}^{m} \sigma_{i\nu}(t)\,dW_\nu(t)\right)$$

- b_i: rate of return of ith stock

- $\sigma_{i\nu}$: sensitivity of the ith stock to νth BM

- Standard assumptions:

$$\int_0^T ||b(t)||\,dt < \infty \text{ for all } T\text{, a.s.}$$

and: there exist $M, \varepsilon > 0$ with

$$\varepsilon||v||^2 \leq v'\sigma(t)(\sigma(t))'v \leq M||v||^2$$

for all fixed $v \in \mathbf{R}^n$ and all t, a.s.

Logarithmic point of view

Set $a_{ij} = \sum_{\nu=1}^{m} \sigma_{i\nu}\sigma_{j\nu} = (\sigma\sigma')_{ij}$.

Use Itô's rule to see that

$$d(\log X_i) = \gamma_i \, dt + \sum_{\nu=1}^{m} \sigma_{i\nu} \, dW_\nu,$$

where $\gamma_i = b_i - \frac{1}{2}a_{ii}$.

γ_i should be interpreted as the logarithmic growth rate of the ith stock.

(Write b_i instead of $b_i(t)$, etc.: everything except constants n,m,ε,M,δ,p are functions of t.)

Portfolios

$\Delta^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \geq 0, \sum x_i = 1\}$.

Portfolio: measurable process $\pi = (\pi_1, \ldots, \pi_n)$ with $\pi \in \Delta^n$ a.s.; π_i represents the proportion of wealth in the ith stock in the portfolio.

Infinitesimal return on X_i is dX_i/X_i, so if Z_π is the value process corresponding to π,

$$\frac{dZ_\pi}{Z_\pi} = \sum_{i=1}^{n} \pi_i \frac{dX_i}{X_i} = \sum_i \pi_i b_i \, dt + \sum_i \sum_\nu \pi_i \sigma_{i\nu} \, dW_\nu.$$

Itô $\Rightarrow d(\log Z_\pi) = \gamma_\pi \, dt + \sum \sum \pi_i \sigma_{i\nu} \, dW_\nu$, where $\gamma_\pi = \sum \pi_i \gamma_i + \gamma_\pi^\star$ and

$$\gamma_\pi^\star = \frac{1}{2} \left(\sum_{i=1}^{n} \pi_i a_{ii} - \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \pi_j a_{ij} \right).$$

$\gamma_\pi^\star$ is the *Excess Growth Rate*, or amount the logarithmic growth rate of portfolio exceeds the weighted average of log growth rates of the component stocks. It equals one half the difference of the weighted average of stock variances and the portfolio variance.

The Market Portfolio

Set $Z = \sum X_i$ and let $\mu_i = X_i/Z$ be cap weights. The vector μ is a portfolio and

$$\frac{dZ_\mu}{Z_\mu} = \sum \mu_i \frac{dX_i}{X_i} = \frac{\sum dX_i}{Z} = \frac{dZ}{Z}.$$

So the portfolio μ mirrors the market and is called the *market portfolio*.

Useful to rank the stocks: set $\mu_{(1)} = \max \mu_i$, and more generally, rank them so that

$$\mu_{(1)} \geq \mu_{(2)} \geq \cdots \geq \mu_{(n)}.$$

(Identities of stocks change with time!)

Diversity

Fix $0 < p < 1$ (not time-dependent). Set

$$D(x) = D_p(x) = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}, x \in \mathbf{R}^n;$$

$$\pi_i^{(p)} = \frac{\mu_i^p}{(D(\mu))^p} = \frac{\mu_i^p}{\mu_1^p + \cdots + \mu_n^p}.$$

This is a *functionally generated portfolio* which is biased towards smaller-cap stocks. Call it a *diversity-weighted portfolio*.

Notions of diversity: say that the market is:

- *diverse* on $[0, T]$ if there exists $\delta > 0$ such that

$$\mu_{(1)}(t) < 1 - \delta \text{ for all } 0 \leq t \leq T, \text{ a.s.}$$

- *weakly diverse* on $[0, T]$ if there exists $\delta > 0$ such that

$$\frac{1}{T}\int_0^T \mu_{(1)}(t)\, dt < 1 - \delta, \text{ a.s.}$$

Arbitrage

Set $\pi = \pi^{(p)}$ (diversity-weighted portfolio). Assume that the market is weakly diverse on $[0, T]$, where T is large enough (we can take $T \geq 2 \log n / p\varepsilon\delta^2$). Then (Fernholz and Karatzas):

$$P[Z_\pi(T) > Z_\mu(T)] = 1.$$

That is, the value of the diversity weighted portfolio at time T is strictly greater than that of the market portfolio, with probability 1.

Also, there is good reason to believe that the diversity-weighted portfolio is no riskier, in the long term, than the market portfolio.

Such a controversial result deserves a proof:

Proof (1)

For portfolio η, quantity $\log(X_i/Z_\eta)$ is relative log return of ith stock v. portfolio. Define τ^η_{ij} by

$$d\langle \log\left(\frac{X_i}{Z_\eta}\right), \log\left(\frac{X_j}{Z_\eta}\right)\rangle_t = \tau^\eta_{ij}\, dt;$$

From above expressions for $d(\log X_i)$ and $d(\log Z_\eta)$, we can easily see that

$$\tau^\eta_{ij} = a_{ij} - \sum_k \eta_k a_{ik} - \sum_k \eta_k a_{jk} + \sum_k \sum_l \eta_k \eta_l a_{kl}.$$

Fact 1: for any portfolios π,η,

$$\gamma^\star_\pi = \frac{1}{2}\left(\sum_i \pi_i \tau^\eta_{ii} - \sum_i \sum_j \pi_i \pi_j \tau^\eta_{ij}\right) \text{ a.s.}$$

Fact 2: for any portfolio π,

$$\sum_i \sum_j \pi_i \pi_j \tau^\pi_{ij} \equiv 0.$$

Fact 3: for any portfolio π,

$$\gamma^\star_\pi = \frac{1}{2} \sum_i \pi_i \tau^\pi_{ii}.$$

Facts 1 and 2 are easy computations based on $\sum_i \pi_i = 1$; they imply Fact 3.

Proof (2)

Set $\eta = \mu$, and write τ_{ij}^{μ} as just τ_{ij}. Since $\mu_i = X_i / Z_\mu$, we have

$$d\langle \log \mu_i, \log \mu_j \rangle_t = \tau_{ij}\, dt.$$

Use Itô on $\mu_i = \exp(\log \mu_i)$ (Fact 4):

$$\begin{aligned} d\mu_i &= \mu_i\, d \log \mu_i + \frac{1}{2} \mu_i\, d\langle \log \mu_i \rangle \\ &= \mu_i\, d \log \mu_i + \frac{1}{2} \mu_i \tau_{ii}\, dt. \end{aligned}$$

It follows that (Fact 5)

$$d\langle \mu_i, \mu_j \rangle_t = \mu_i \mu_j d\langle \log \mu_i, \log \mu_j \rangle_t = \mu_i \mu_j \tau_{ij}\, dt.$$

Sum this last relation over j to get

$$d\langle \mu_i, \sum_j \mu_j \rangle_t = \mu_i \sum_j \mu_j \tau_{ij}\, dt.$$

Since $\sum_j \mu_j = 1$, LHS is 0. So (Fact 6)

$$\sum_j \mu_j \tau_{ij} \equiv 0 \quad \text{a.s. for all } j.$$

Proof (3): Portfolio Generating Functions (PGFs)

Theorem (Fernholz): Let S be a positive C^2 function defined on a neighbourhood of Δ^n. Assume $x_i \partial_i \log S(x)$ is bounded on Δ^n. Let

$$\pi_i = \left((\partial_i \log S)(\mu) + 1 - \sum_j (\partial_j \log S)(\mu) \right) \mu_i.$$

Then

$$d \log \left(\frac{Z_\pi}{Z_\mu} \right) = d \log S(\mu) + d\Theta,$$

where $d\Theta$ is the *drift term*:

$$d\Theta = -\frac{1}{2S(\mu)} \sum_i \sum_j \partial_{ij} S(\mu) \mu_i \mu_j \tau_{ij} \, dt.$$

We say that S *generates* the portfolio π. In particular, we will set $S = D_p$ for the diversity-weighted portfolio.

We'll prove this theorem first, then return to the proof of the main theorem.

Proof of PGF Theorem

From expressions for $d \log Z_\pi$ and $d \log X_i$,

$$d \log Z_\pi = \sum_i \pi_i \, d \log X_i + \gamma^\star_\pi.$$

So we have:

$$\begin{aligned}
d \log \left(\frac{Z_\pi}{Z_\mu} \right) &= \sum_i \pi_i \, d \log(X_i / Z_\mu) + \gamma^\star_\pi \\
&= \sum_i \pi_i \, d \log \mu_i + \gamma^\star_\pi \\
\text{(Fact 4)} \quad &= \sum_i \pi_i \left(\frac{d\mu_i}{\mu_i} - \frac{1}{2} \tau_{ii} \, dt \right) \\
\text{(Fact 1)} \quad & \quad + \frac{1}{2} \left(\sum_i \pi_i \tau_{ii} - \sum_i \sum_j \pi_i \pi_j \tau_{ij} \right) dt \\
&= \sum_i \frac{\pi_i}{\mu_i} \, d\mu_i - \frac{1}{2} \sum_i \sum_j \pi_i \pi_j \tau_{ij} \, dt \\
&= (\mathbf{I}) - \frac{1}{2} (\mathbf{II}) \, dt.
\end{aligned}$$

Set $\varphi(t) = 1 - \sum_j \mu_j (\partial_j \log S)(\mu)$, so that

$$\frac{\pi_i}{\mu_i} = (\partial_i \log S)(\mu) + \varphi.$$

Proof of PGF Theorem (2)

$$\begin{aligned}\textbf{(I)} &= \sum_i \frac{\pi_i}{\mu_i} d\mu_i = \sum_i (\partial_i \log S)(\mu)\, d\mu_i + \varphi \sum_i d\mu_i \\ &= \sum_i (\partial_i \log S)(\mu)\, d\mu_i.\end{aligned}$$

Here we used $\sum_i \mu_i \equiv 1 \Rightarrow \sum_i d\mu_i \equiv 0$.

$$\begin{aligned}\textbf{(II)} &= \sum_i \sum_j \pi_i \pi_j \tau_{ij} \\ &= \sum_i \sum_j (\partial_i \log S(\mu))(\partial_j \log S(\mu)) \mu_i \mu_j \tau_{ij} \\ &\quad +2\varphi \sum_i \sum_j (\partial_i \log S(\mu)) \mu_i \mu_j \tau_{ij} \\ &\quad +\varphi^2 \sum_i \sum_j \mu_i \mu_j \tau_{ij} \\ &= \sum_i \sum_j (\partial_i \log S(\mu))(\partial_j \log S(\mu)) \mu_i \mu_j \tau_{ij}.\end{aligned}$$

(We used $\tau_{ij} = \tau_{ji}$ and Fact 6.) Calculus:

$$\partial_{ij} \log S = \frac{\partial_{ij} S}{S} - (\partial_i \log S)(\partial_j \log S); \text{ so}$$

$$\textbf{(II)} = \sum_i \sum_j \left(\frac{\partial_{ij} S(\mu)}{S(\mu)} - \partial_{ij} \log S(\mu) \right) \mu_i \mu_j \tau_{ij}.$$

Proof of PGF Theorem (3)

So we see that

$$d \log \left(\frac{Z_\pi}{Z_\mu} \right) = \sum_i (\partial_i \log S)(\mu)\, d\mu_i$$
$$- \frac{1}{2} \sum_i \sum_j \frac{\partial_{ij} S(\mu)}{S(\mu)} \mu_i \mu_j \tau_{ij}\, dt$$
$$+ \frac{1}{2} \sum_i \sum_j (\partial_{ij} \log S)(\mu) \mu_i \mu_j \tau_{ij}\, dt.$$

Use Itô's Rule and Fact 5 to see that

$$d \log S(\mu) = \sum_i (\partial_i \log S)(\mu)\, d\mu_i$$
$$+ \frac{1}{2} \sum_i \sum_j (\partial_{ij} \log S)(\mu) \mu_i \mu_j \tau_{ij}\, dt.$$

This finishes the proof. Next, apply theorem with $S = D_p$.

PGF Theorem and Diversity

$$\log D = \frac{1}{p}\log(x_1^p + \cdots + x_n^p) = \frac{1}{p}\log(D^p), \text{ so}$$

$$\partial_i \log D(x) = \frac{1}{p} p x_i^{p-1} \frac{1}{D(x)^p} = \frac{x_i^{p-1}}{D(x)^p}.$$

$$\mu_i \partial_i \log D(\mu) = \frac{\mu_i^p}{(D(\mu))^p} = \pi_i^{(p)}.$$

It easily follows that D generates $\pi^{(p)}$. Also:

$$\partial_i D(x) = x_i^{p-1}(D(x))^{1-p}$$

$$\begin{aligned}\partial_{ij} D(x) &= (1-p)x_i^{p-1}x_j^{p-1}(D(x))^{2-p} \\ &\quad -\delta_{ij}(1-p)x_i^{p-2}(D(x))^{1-p}; \text{ so:}\end{aligned}$$

$$\begin{aligned}&-\frac{1}{2D(\mu)}\sum_i\sum_j \partial ij D(\mu)\mu_i\mu_j\tau_{ij} \\ &= \frac{1}{2}(1-p)\Big(\sum_i \mu_i^p D(\mu)^{-p}\tau_{ii} \\ &\quad -\sum_i\sum_j \mu_i^p\mu_j^p D(\mu)^{-2p}\tau_{ij}\Big) \\ &= \frac{1}{2}(1-p)\Big(\sum_i \pi_i\tau_{ii} - \sum_i\sum_j \pi_i\pi_j\tau_{ij}\Big)\end{aligned}$$

Proof of the Main Theorem(4)

This shows that $d\Theta = (1-p)\gamma_\pi^\star$. So

$$d\log(Z_\pi/Z_\mu) = d\log D(\mu) + (1-p)\gamma_\pi^\star\, dt, \text{ or}$$

$$\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) = \log\left(\frac{D(\mu(T))}{D(\mu(0))}\right) + (1-p)\int_0^T \gamma_\pi^\star(t)\, dt.$$

We want LHS to be positive a.s. Now

$$1 = \sum_i \mu_i \le \sum_i \mu_i^p = D(\mu)^p \le n^{1-p}$$

(equality on RHS when $\mu_i = \frac{1}{n}$ for all i) so

$$0 \le p\log D(\mu) \le (1-p)\log n.$$

$$\begin{aligned} \log\left(\frac{D(\mu(T))}{D(\mu(0))}\right) &= \log D(\mu(T)) - \log D(\mu(0)) \\ &\ge -\frac{1-p}{p}\log n. \end{aligned}$$

Now only need to estimate $\int_0^T \gamma_\pi^\star(t)\, dt$.

Proof (5) - Estimating $\gamma^{\star}_{\pi}(t)$

Step 1: $\max \pi_j \leq \max \mu_j = \mu_{(1)}$ for all t, a.s. To see this, for each i, $\mu_i^{1-p} \leq \mu_{(1)}^{1-p}$. Since $\sum_i \mu_i \equiv 1$,

$$\mu_{(1)}^p = \sum_i \mu_i \mu_{(1)}^p \leq \sum_i \mu_i^p \mu_{(1)} = \mu_{(1)} \sum_i \mu_i^p; \text{ so}$$

$$\max \pi_j = \frac{\mu_{(1)}^p}{\sum_i \mu_i^p} \leq \mu_{(1)}.$$

Step 2: $\tau_{ii}^{\pi} \geq \varepsilon(1-\pi_i)^2$ for all t, a.s. To see this, let $v = (\pi_1, \ldots, \pi_i - 1, \ldots, \pi_n)$. By easy calculation and assumption:

$$\tau_{ii}^{\pi} = v' \sigma \sigma' v \geq \varepsilon ||v||^2 \geq \varepsilon (1-\pi_i)^2.$$

Step 3: estimate $\gamma^{\star}_{\pi}$ using Fact 3, $\sum_i \pi_i \equiv 1$ and Steps 1 and 2:

$$\begin{aligned} \gamma^{\star}_{\pi} &= \frac{1}{2} \sum_i \pi_i \tau_{ii}^{\pi} \geq \frac{\varepsilon}{2} \sum_i (1-\pi_i)^2 \pi_i \\ &\geq \frac{\varepsilon}{2} (1 - \max \pi_i)^2 \sum_i \pi_i \geq \frac{\varepsilon}{2} (1 - \mu_{(1)})^2. \end{aligned}$$

Proof (6) - conclusion

$$\int_0^T \gamma_\pi^\star(t)\,dt \geq \frac{\varepsilon}{2}\int_0^T (1-\mu_{(1)})^2\,dt$$
$$\geq \frac{\varepsilon}{2T}\left(\int_0^T (1-\mu_{(1)})\,dt\right)^2 > \frac{\varepsilon}{2}\delta^2 T.$$

(Used Cauchy-Schwarz and weak diversity.)
Putting it all together,

$$\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) = \log\left(\frac{D(\mu(T))}{D(\mu(0))}\right)$$
$$+(1-p)\int_0^T \gamma_\pi^\star(t)\,dt$$
$$> -\frac{1-p}{p}\log n + (1-p)\frac{\varepsilon}{2}\delta^2 T$$
$$= (1-p)\left(\frac{\varepsilon T\delta^2}{2} - \frac{\log n}{p}\right)$$
$$\geq 0 \text{ if } T \geq \frac{2\log n}{p\varepsilon\delta^2}, \text{ a.s.}$$

It follows that $Z_\pi(T) > Z_\mu(T)$ a.s. as claimed.

Risk and Reality

Consider risk of diversity-weighted portfolio. Graph of $D(\mu)$ for $p = \frac{1}{2}$ shows that it is mean-reverting with relaxation period of about 10 years. So, if we sample Z_π and Z_μ with frequency $>$ 10 years, sample variance should be about the same.

In reality, rebalance every month. In periods of stable diversity $D(\mu)$, the diversity-weighted portfolio (with $p = 0.76$) should outperform the market portfolio (S&P 500) by about 0.5% per year. Turnover in former is about 12% per year, compared with 6% for S&P portfolio or 8% for Russell 1000. This isn't significant.

Conclusion: diversity-weighted index seems to be superior to cap-weighted index.

Ranked Portfolio Generating Functions

Recall ranked market weights:

$$\mu_{(1)} \geq \mu_{(2)} \geq \cdots \geq \mu_{(n)}.$$

Define random permutation p_t of $\{1, \ldots, n\}$ by

$$\mu_{p_t(k)}(t) = \mu_{(k)}(t) \text{ and}$$

$$p_t(k) < p_t(k+1) \text{ if } \mu_{(k)}(t) = \mu_{(k+1)}(t).$$

So $p_t(k)$ is the number of the kth-ranked stock at time t. Now suppose that $\mathbf{S}$ is a function defined on a neighbourood of Δ^n which is symmetric in all variables. Think of $\mathbf{S}$ as a function of rank:

$$\mathbf{S}(x_1, \ldots, x_n) = S(x_{(1)}, \ldots, x_{(n)})$$

for some smooth function S. Also assume that no three stock prices are ever equal, and that times any two stock prices are equal are of measure 0 a.s. Then have a different version of PGF theorem:

Ranked PGF Theorem

Define portfolio by

$$\pi_{p_t(k)} = \Big((\partial_i \log S)(\mu_{(\cdot)}) + 1 - \sum_j (\partial_j \log S)(\mu_{(\cdot)})\Big)\mu_{(k)}.$$

Then

$$d\log\left(\frac{Z_\pi}{Z_\mu}\right) = d\log \mathbf{S}(\mu) + d\Theta,$$

where the drift term $d\Theta$ is now given by:

$$d\Theta = -\frac{1}{2\mathbf{S}(\mu)}\sum_i\sum_j \partial_{ij}S(\mu)\mu_{(i)}\mu_{(j)}\tau_{(ij)}\,dt + \frac{1}{2}\sum_{k=1}^{n-1}(\pi_{p_t(k+1)} - \pi_{p_t(k)})\,d\Lambda_{\log\mu_{(k)}-\log\mu_{(k+1)}}.$$

Here $\tau_{(ij)}$ is shorthand for $\tau_{p_t(i)p_t(j)}$ and Λ is *local time*. (Recall that

$$\Lambda_X(t) = \frac{1}{2}\left(|X(t)| - |X(0)| - \int_0^t \mathrm{sgn}(X(s))\,dX(s)\right)$$

for the continuous semimartingale X.) The proof of this is messy but similar to previous result. The local times arise from when stocks switch rank.

Fix $m < n$; let $\mathbf{S}_L(x) = x_{(1)} + \cdots + x_{(m)}$, so that $S(x) = x_1 + \cdots + x_m$. In context of above theorem, set

$$\begin{aligned}\xi_{p_t(k)} &= \Big((\partial_i \log S)(\mu_{(\cdot)}) + 1 \\ &\qquad - \sum_j (\partial_j \log S)(\mu_{(\cdot)})\Big)\mu_{(k)}\end{aligned}$$

which simplifies to $\xi_{p_t(k)} = \mu_{(k)}/\mathbf{S}_L(\mu)$ if $k \leq m$ and $\xi_{p_t(k)} = 0$ otherwise. This portfolio is cap-weighted in m largest stocks. Now apply theorem; since $\partial_{ij} S \equiv 0$, only part of $d\Theta$ is local time part. Standard result:

$$I_{\{0\}}(X(t))\, d\Lambda_X(t) = d\Lambda_X(t) \text{ a.s.}$$

(local time measure is supported in the null set $\{X(t) = 0\}$). So if $\xi_{p_t(k+1)} \neq \xi_{p_t(k)}$, then $d\Lambda_{\log\mu_{(k)} - \log\mu_{(k+1)}} = 0$. Only term contributing to sum

$$d\Theta = \frac{1}{2}\sum_{k=1}^{n-1} (\xi_{p_t(k+1)} - \xi_{p_t(k)})\, d\Lambda_{\log\mu_{(k)} - \log\mu_{(k+1)}}$$

is when $k = m$. So,

The Size Effect (2)

$$d\log\left(\frac{Z_\xi}{Z_\mu}\right) = d\log \mathbf{S}_L(\mu) - \frac{1}{2}\xi_{(m)}\,d\Lambda_{\log\mu_{(m)}-\log\mu_{(m+1)}}.$$

Now define $\mathbf{S}_S(x) = x_{(m+1)} + \cdots + x_{(n)}$ and repeat previous argument to get portfolio η of $(n-m)$th smallest stocks, cap-weighted:

$$d\log\left(\frac{Z_\eta}{Z_\mu}\right) = d\log \mathbf{S}_S(\mu) + \frac{1}{2}\eta_{(m+1)}\,d\Lambda_{\log\mu_{(m)}-\log\mu_{(m+1)}}.$$

Using both equations,

$$d\log\left(\frac{Z_\eta}{Z_\xi}\right) = d\log\left(\frac{\mathbf{S}_S(\mu)}{\mathbf{S}_L(\mu)}\right) + \frac{1}{2}(\xi_{(m)} + \eta_{(m+1)})\,d\Lambda_{\log\mu_{(m)}-\log\mu_{(m+1)}}.$$

Suppose relative caps of small and large stocks are stable over time. Then drift term dominates and is strictly increasing. So, in a stable market, return of small stocks is greater than return of large stocks regardless of risk-level.