An Exact and Explicit Solution for the Value of American Put and its Optimal Exercise Boundary

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Outline

- **Background**
- **An exact and explicit solution for American put options**
- **Examples and Discussions**
- **Concluding Remarks**

Background

- **American Options and Their Optimal Exercise Boundary**
  - Analytical Approximation Methods;
  - Numerical Solutions;

Numerical Approaches:

- **Directly solving PDE:**
  - Brennan and Schwartz (1977), Wu and Kwok (1997): Finite Difference Method (FDM);
  - Allegretto *et al.* (2001): Finite Element Method (FEM);
  - Hon and Mao (1997): Radial Basis Function Method (RBF).

- **Risk Neutral Approaches**
  - Cox *et al.* (1979): The Binomial Method
  - Grant *et al.* (1996): The Monte Carlo simulation method
  - Longstaff and Schwartz (2001): the Least Squares Method

Analytical Approximations:

- Geske and Johnson (1984):the compound-option approximation method;
- MacMillan (1986): the quadratic approximation method;
- Johnson (1983): the interpolation method;
- Huang *et al.* (1996), Ju (1998): the integral-equation method;
- Bunch and Johnson (2000): Algebraic Equation Method;
- Zhu (2004): Laplace Transform based on the pseudo-steady-state approximation.

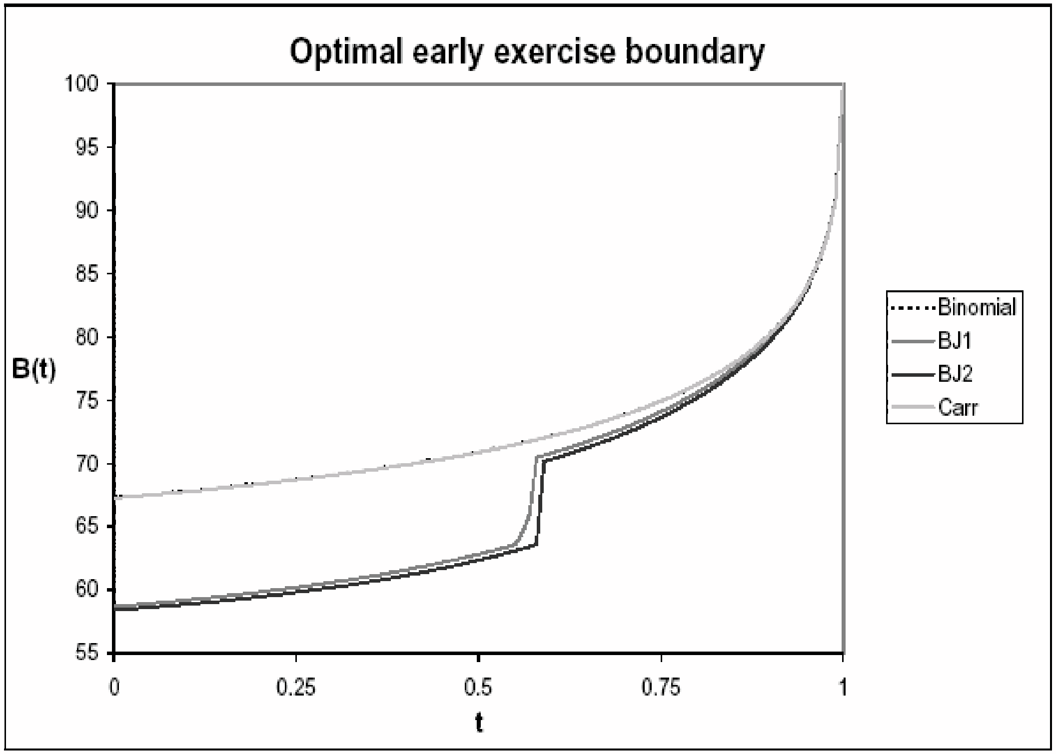

Figure 1: A. Basso, M. Nardon and P. Pianca's 2002 working paper

The PDE system for pricing American Put Options and determining the optimal exercise boundary

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + rS\dfrac{\partial V}{\partial S} - rV = 0, \\
V(S_f(t), t) = X - S_f(t), \\
\dfrac{\partial V}{\partial S}(S_f(t), t) = -1, \\
\lim\limits_{S\to\infty} V(S, t) = 0, \\
V(S, T) = \max\{X - S, 0\}.
\end{cases}
$$

The dimensionless PDE system becomes:

$$\begin{cases} -\dfrac{\partial V}{\partial \tau} + S^2 \dfrac{\partial^2 V}{\partial S^2} + \gamma S \dfrac{\partial V}{\partial S} - \gamma V = 0, \\ V(S_f(\tau), \tau) = 1 - S_f(\tau), \\ \dfrac{\partial V}{\partial S}(S_f(\tau), \tau) = -1, \\ \lim_{S \to \infty} V(S, \tau) = 0, \\ V(S, 0) = \max\{1 - S, 0\}, \end{cases}$$

in which $\gamma \equiv \dfrac{2r}{\sigma^2}$ **can be viewed as an interest rate relative to the volatility of the asset price.**

An Exact Solution in Series Expansion Form

We begin with introducing the Landau transform

$$x = \ln \frac{S}{S_f(\tau)}.$$

An Exact Solution in Series Expansion Form

The PDE system under the Landau transform becomes

$$
* \begin{cases}
\dfrac{\partial V}{\partial \tau} - \dfrac{\partial^2 V}{\partial x^2} - (\gamma - 1)\dfrac{\partial V}{\partial x} + \gamma V = \dfrac{1}{S_f(\tau)} \dfrac{dS_f}{d\tau} \dfrac{\partial V}{\partial x}, \\
V(x, 0) = 0, \\
V(0, \tau) = 1 - S_f(\tau), \\
\dfrac{\partial V}{\partial x}(0, \tau) = -S_f(\tau), \\
\lim\limits_{x \to \infty} V(x, \tau) = 0.
\end{cases}
$$

The homotopy-analysis method

- Ortega and Rheinboldt (1970): Solving nonlinear algebraic equations;

- Liao (1997) and Liao and Zhu (1999): Solving heat transfer problems;

- Liao and Zhu (1996) and Liao and Campo (2002): Solving fluid flow problems.

The essential concept of the method is to construct a continuous "homotopic deformation" through a series expansion of the unknown function.

An Exact Solution in Series Expansion Form

Let's now construct two new unknown functions $\bar{V}(x,\tau,p)$ and $\bar{S}_f(\tau,p)$ that satisfy the following differential system,

$$\begin{cases} (1-p)\mathcal{L}[\bar{V}(x,\tau,p) - \bar{V}_0(x,\tau)] = -p\{\mathcal{A}[\bar{V}(x,t,p), \bar{S}_f(\tau,p)]\}, \\ \bar{V}(x,0,p) = (1-p)\bar{V}_0(x,0), \\ \bar{V}(0,\tau,p) + \bar{S}_f(\tau,p) = 1, \\ \dfrac{\partial \bar{V}}{\partial x}(0,\tau,p) + \bar{S}_f(\tau,p) = (1-p)\left[1 + \dfrac{\partial \bar{V}_0}{\partial x}(0,\tau) - \bar{V}_0(0,\tau)\right], \\ \lim\limits_{x\to\infty} \bar{V}(x,\tau,p) = 0, \end{cases}$$

where $\mathcal{L}$ is a differential operator defined as

$$\mathcal{L} = \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2} - (\gamma - 1)\frac{\partial}{\partial x} + \gamma,$$

and $\mathcal{A}$ is a functional defined as

$$\mathcal{A}[\bar{V}(x,\tau,p), \bar{S_f}(\tau,p)] = \mathcal{L}(\bar{V}) - \frac{1}{\bar{S_f}(\tau,p)}\frac{\partial \bar{S_f}}{\partial \tau}(\tau,p)\frac{\partial \bar{V}}{\partial x}(x,\tau,p).$$

An Exact Solution in Series Expansion Form

With $p = 0$, we have, from the differential system *

$$
\begin{cases}
\mathcal{L}[\bar{V}(x,\tau,0)] = \mathcal{L}[\bar{V}_0(x,\tau)], \\[2ex]
\bar{V}(x,0,0) = \bar{V}_0(x,0), \\[2ex]
\bar{V}(0,\tau,0) + \bar{S}_f(\tau,0) = 1, \\[2ex]
\dfrac{\partial \bar{V}}{\partial x}(0,\tau,0) + \bar{S}_f(\tau,0) = 1 + \dfrac{\partial \bar{V}_0}{\partial x}(0,\tau) - \bar{V}_0(0,\tau), \\[2ex]
\lim\limits_{x\to\infty} \bar{V}(x,\tau,0) = 0.
\end{cases}
$$

An Exact Solution in Series Expansion Form

Clearly, the solution of this differential system is

$$\begin{cases} \bar{V}(x,\tau,0) = \bar{V}_0(x,\tau), \\ \\ \bar{S}_f(\tau,0) = 1 - \bar{V}_0(0,\tau) = \bar{S}_0(\tau), \end{cases}$$

so long as the initial guess $\bar{V}_0(x,\tau)$ satisfies the condition

$$\lim_{x\to\infty} \bar{V}_0(x,\tau) = 0.$$

An Exact Solution in Series Expansion Form

When $p = 1$, the differential system * becomes

$$
\begin{cases}
\mathcal{L}[\bar{V}(x,\tau,1)] = \dfrac{1}{\bar{S}_f(\tau,1)} \dfrac{\partial \bar{S}_f}{\partial \tau}(\tau,1) \dfrac{\partial \bar{V}}{\partial x}(x,\tau,1), \\[2ex]
\bar{V}(x,0,1) = 0, \\[2ex]
\bar{V}(0,\tau,1) = 1 - \bar{S}_f(\tau,1), \\[2ex]
\dfrac{\partial \bar{V}}{\partial x}(0,\tau,1) = -\bar{S}_f(\tau,1), \\[2ex]
\lim\limits_{x \to \infty} \bar{V}(x,\tau,1) = 0.
\end{cases}
$$

An Exact Solution in Series Expansion Form

Comparing this PDE and the original one we were trying to solve, it is obvious that the solution we seek is nothing but

$$\begin{cases} V(x,\tau) = \bar{V}(x,\tau,1), \\ \\ S_f(\tau) = \bar{S_f}(\tau,1). \end{cases}$$

To find the values of $\bar{V}(x,\tau,1)$ and $\bar{S_f}(\tau,1)$, we can now expand the functions $\bar{V}(x,\tau,p)$ and $\bar{S_f}(\tau,p)$ as a Taylor's series expansion of p

$$\begin{aligned}\bar{V}(x,\tau,p) &= \sum_{m=0}^{\infty} \frac{\bar{V}_m(x,\tau)}{m!} p^m, \\ \bar{S_f}(\tau,p) &= \sum_{m=0}^{\infty} \frac{\bar{S}_m(\tau)}{m!} p^m,\end{aligned}$$

where $\bar{V}_m$ is the mth-order partial derivative of $\bar{V}(x,\tau,p)$ with respect to p and then evaluated at $p=0$,

$$\bar{V}_m(x,\tau) = \left.\frac{\partial^m}{\partial p^m}\bar{V}(x,\tau,p)\right|_{p=0},$$

and $\bar{S}_m$ is the mth-order partial derivative of $\bar{S}(\tau,p)$ with respect to p and then evaluated at $p=0$,

$$\bar{S}_m(\tau) = \left.\frac{\partial^m}{\partial p^m}\bar{S_f}(\tau,p)\right|_{p=0}.$$

An Exact Solution in Series Expansion Form

To find all the coefficients in the above Taylor's expansions, we need to derive a set of governing partial differential equations and appropriate boundary and initial conditions for the unknown functions $\bar{V}_m(x, \tau)$ and $\bar{S}_m(\tau)$. They can be derived from differentiating each equation in the differential system * with respect to p and then setting p equal to zero.

The 1st order differential system:

$$\begin{cases} \mathcal{L}[\bar{V}_1(x,\tau)] = -\mathcal{L}[\bar{V}_0(x,\tau)] + \mathcal{A}^{'}(x,\tau,0), \\ \bar{V}_1(x,0) = -\bar{V}_0(x,0), \\ \bar{V}_1(0,\tau) + \bar{S}_1(\tau) = 0, \\ \dfrac{\partial \bar{V}_1}{\partial x}(0,\tau) + \bar{S}_1(\tau) = \bar{V}_0(0,\tau) - \dfrac{\partial \bar{V}_0}{\partial x}(0,\tau) - 1, \\ \lim_{x\to\infty} \bar{V}_1(x,\tau) = 0, \end{cases}$$

The nth order differential system:

$$\begin{cases} \mathcal{L}[\bar{V}_n(x,\tau)] = n \dfrac{\partial^{n-1}\mathcal{A}^{'}}{\partial^{n-1}p}\bigg|_{p=0}, \\ \bar{V}_n(x,0) = 0, \\ \bar{V}_n(0,\tau) + \bar{S}_n(\tau) = 0, \qquad\qquad \text{if} \quad n \geq 2, \\ \dfrac{\partial \bar{V}_n}{\partial x}(0,\tau) + \bar{S}_n(\tau) = 0, \\ \lim\limits_{x\to\infty} \bar{V}_n(x,\tau) = 0, \end{cases}$$

where

$$\mathcal{A}^{'}(x,\tau,p) = \frac{1}{\bar{S}_f(\tau,p)} \frac{\partial \bar{S}_f}{\partial \tau}(\tau,p) \frac{\partial \bar{V}}{\partial x}(x,\tau,p).$$

After eliminating $\bar{S}_n(\tau)$ from the two boundary conditions at $x = 0$ in above two equations, we can rewrite them in a general form

$$\begin{cases} \mathcal{L}[\bar{V}_n(x,\tau)] = f_n(x,\tau), \\ \\ \bar{V}_n(x,0) = \psi_n(x), \\ \\ \dfrac{\partial \bar{V}_n}{\partial x}(0,\tau) - \bar{V}_n(0,\tau) = \phi_n(\tau), \\ \\ \bar{V}_n(\infty,\tau) = 0, \end{cases}$$

with $f_n(x,\tau)$, $\psi_n(x)$ and $\phi_n(\tau)$ being expressed respectively as

$$f_n(x,\tau) = \begin{cases} -\mathcal{L}[\bar{V}_0(x,\tau)] + \mathcal{A}^{'}(x,\tau,0), & \text{if} \quad n = 1, \\ n \dfrac{\partial^{n-1}\mathcal{A}^{'}}{\partial p^{n-1}}\Big|_{p=0}, & \text{if} \quad n \geq 2, \end{cases}$$

$$\psi_n(x) = \begin{cases} -\bar{V}_0(x,0), & \text{if} \quad n = 1, \\ 0, & \text{if} \quad n \geq 2, \end{cases}$$

$$\phi_n(\tau) = \begin{cases} \bar{V}_0(0,\tau) - \dfrac{\partial \bar{V}_0}{\partial x}(0,\tau) - 1, & \text{if} \quad n = 1, \\ 0, & \text{if} \quad n \geq 2. \end{cases}$$

$$
\begin{aligned}
\bar{V}_n(x,\tau) \quad = \quad & e^{-\frac{1}{2}(\gamma-1)x-\frac{1}{4}(\gamma+1)^2\tau}\left\{\frac{1}{\sqrt{\pi}}\left\{e^{\frac{1}{2}(\gamma-1)x}\int_{-\frac{x}{2\sqrt{\tau}}}^{\frac{x}{2\sqrt{\tau}}}\psi_n(2\sqrt{\tau}\,\xi+x)\right.\right. \\
& \cdot e^{(\gamma-1)\sqrt{\tau}\xi-\xi^2}\,d\xi+\int_{\frac{x}{2\sqrt{\tau}}}^{\infty}\left[e^{\frac{1}{2}(\gamma-1)x}\psi_n(2\sqrt{\tau}\,\xi+x)\right. \\
& \left.\left.+e^{-\frac{1}{2}(\gamma-1)x}\psi_n(2\sqrt{\tau}\,\xi-x)\right]e^{(\gamma-1)\sqrt{\tau}\xi-\xi^2}d\xi\right\} \\
& -(\gamma+1)\sqrt{\tau}\,e^{-\frac{1}{2}(\gamma-1)x+\frac{(\gamma+1)^2}{4}\tau} \\
& \cdot\int_{\frac{x}{2\sqrt{\tau}}}^{\infty}\psi_n(2\sqrt{\tau}\xi\,-x)e^{2\gamma\sqrt{\tau}\,\xi}\cdot\ \mathrm{erfc}(\xi+\frac{(\gamma+1)}{2}\sqrt{\tau})d\xi \\
& -\frac{2}{\sqrt{\pi}}e^{\frac{(\gamma+1)^2}{4}\tau}\int_0^{\infty}e^{-\frac{(\gamma+1)}{2}\eta}\int_{\frac{x+\eta}{2\sqrt{\tau}}}^{\infty}\phi_n\left(\tau-\frac{(x+\eta)^2}{4\xi^2}\right) \\
& \cdot e^{-\left[\frac{(\gamma+1)(x+\eta)}{4\xi}\right]^2-\xi^2}d\xi d\eta
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{\tau}\left\{\frac{e^{\frac{(\gamma+1)^2}{4}\eta}}{\sqrt{\pi}}\left[e^{\frac{1}{2}(\gamma-1)x}\int_{-\frac{x}{2\sqrt{\tau-\eta}}}^{\frac{x}{2\sqrt{\tau-\eta}}} f_n(2\sqrt{\tau-\eta}\,\xi+x,\,\eta)\right.\right.\\
&e^{(\gamma-1)\sqrt{\tau-\eta}\,\xi-\xi^2}d\xi\\
&+\int_{\frac{x}{2\sqrt{\tau-\eta}}}^{\infty}\left[e^{\frac{1}{2}(\gamma-1)x}f_n(2\sqrt{\tau-\eta}\,\xi+x,\,\eta)\right.\\
&\left.\left.+e^{-\frac{1}{2}(\gamma-1)x}f_n(2\sqrt{\tau-\eta}\,\xi-x,\eta)\right]e^{(\gamma-1)\sqrt{\tau-\eta}\,\xi-\xi^2}d\xi\right]\\
&-(\gamma+1)\sqrt{\tau-\eta}e^{-\frac{1}{2}(\gamma-1)x+\frac{(\gamma+1)^2}{4}\tau}\int_{\frac{x}{2\sqrt{\tau-\eta}}}^{\infty} f_n(2\sqrt{\tau-\eta}\,\xi-x,\eta)\\
&\left.\left.\cdot\ e^{2\gamma\sqrt{\tau-\eta}\ \xi}\ \mathrm{erfc}(\xi+\frac{(\gamma+1)}{2}\sqrt{\tau-\eta})d\xi\right\}d\eta\right\},
\end{aligned}
$$

Initial Guess

Choosing the corresponding European option value as the initial guess yields the following three apparent merits:

- **The boundary condition at $x = \infty$ is automatically satisfied**

- $f_1(x, \tau)$ **is further simplified because the first term on the righthand side vanishes**

- $\psi_1(x)$ **vanishes as well, so the integral involving** ψ_n **is entirely eliminated**

Optimal Exercise Price

$$
\begin{aligned}
S_f(\tau) \quad = \quad & \frac{2}{\sqrt{\pi}} e^{-\frac{1}{2}(\gamma-1)x-\frac{1}{4}(\gamma+1)^2\tau} \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ e^{\frac{(\gamma+1)^2}{4}\tau} \int_0^{\infty} e^{-\frac{(\gamma+1)}{2}\eta} \right. \\
& \int_{\frac{\eta}{2\sqrt{\tau}}}^{\infty} \phi_n\left(\tau - \frac{\eta^2}{4\xi^2}\right) e^{-\left[\frac{(\gamma+1)\eta}{4\xi}\right]^2-\xi^2} d\xi d\eta \\
& + \int_0^{\tau} \left[e^{\frac{(\gamma+1)^2}{4}\eta} \int_0^{\infty} f_n(2\sqrt{\tau-\eta}\,\xi,\eta) e^{(\gamma-1)\sqrt{\tau-\eta}\,\xi-\xi^2} d\xi \right. \\
& -\frac{\sqrt{\pi}}{2}(\gamma+1)\sqrt{\tau-\eta}\, e^{\frac{(\gamma+1)^2}{4}\tau} \int_0^{\infty} f_n(2\sqrt{\tau-\eta}\,\xi,\eta) \\
& \left.\left. \cdot\; e^{2\gamma\sqrt{\tau-\eta}\;\xi}\ \mathrm{erfc}\left(\xi + \frac{(\gamma+1)}{2}\sqrt{\tau-\eta}\right) d\xi \right] d\eta \right\}
\end{aligned}
$$

Convergence of the series

The convergence criterion is

$$\lim_{m\to\infty} (\frac{m}{m+1}) |\frac{\bar{V}_{m+1}}{\bar{V}_m}| p < 1,$$

for $p \in [0, 1]$, according to the d'Alembert's ratio test.

This is thus equivalent to

$$\lim_{m\to\infty} |\frac{\bar{V}_{m+1}}{\bar{V}_m}| < 1.$$

Numerical evidence will be provided to support the satisfaction of this convergence criterion.

Verification Through Examples:

Example 1:

Use the same example used in Zhu (2004):

- **Strike price** $X = \$100$,
- **Risk-free interest rate** $r = 0.1$,
- **Volatility** $\sigma = 0.3$,
- **Time to expiry** $T = 1$ **(year).**

In terms of the dimensionless variables, the two parameters involved are $\gamma = 2.2222$ **and** $\tau_{exp} = 0.045$.

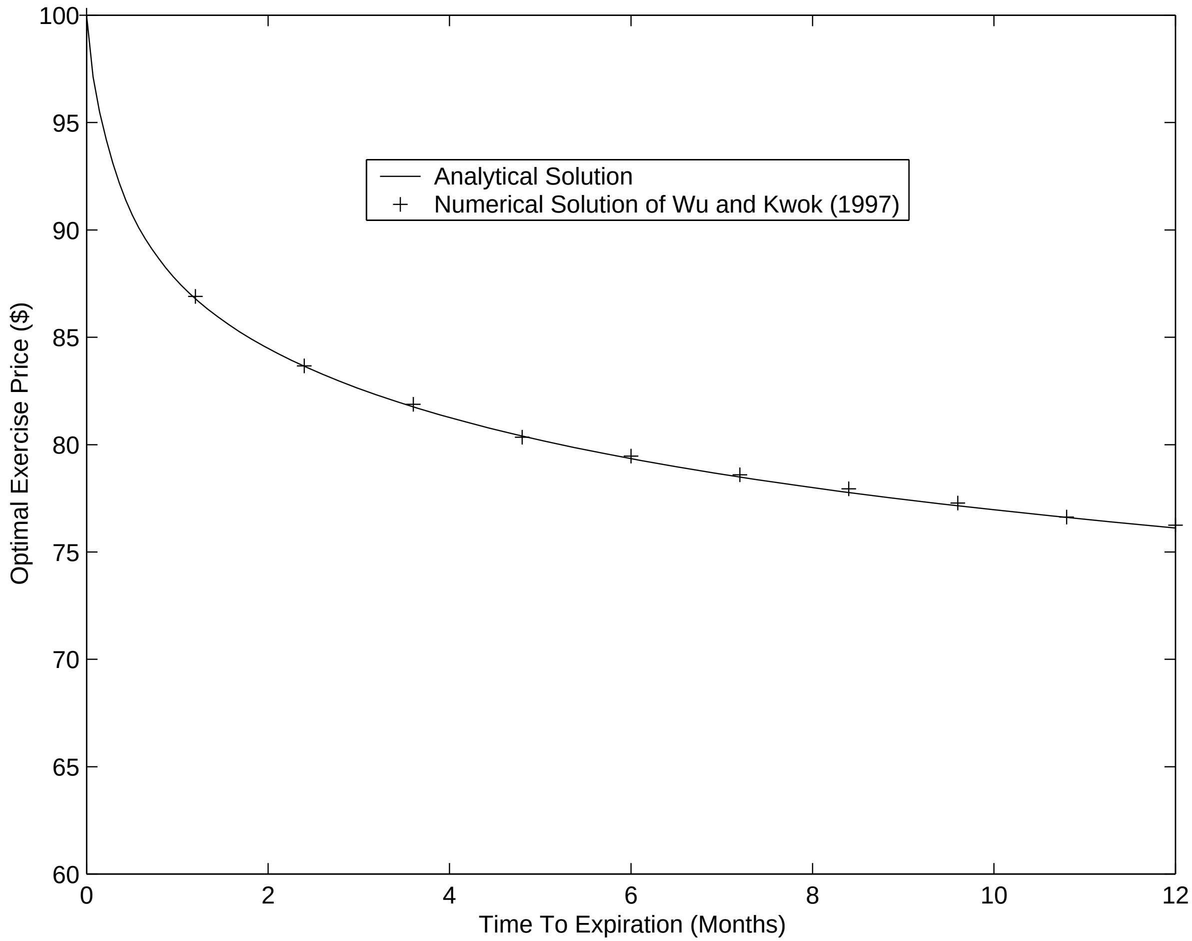

Figure 2: Optimal exercise prices for the case in Example 1

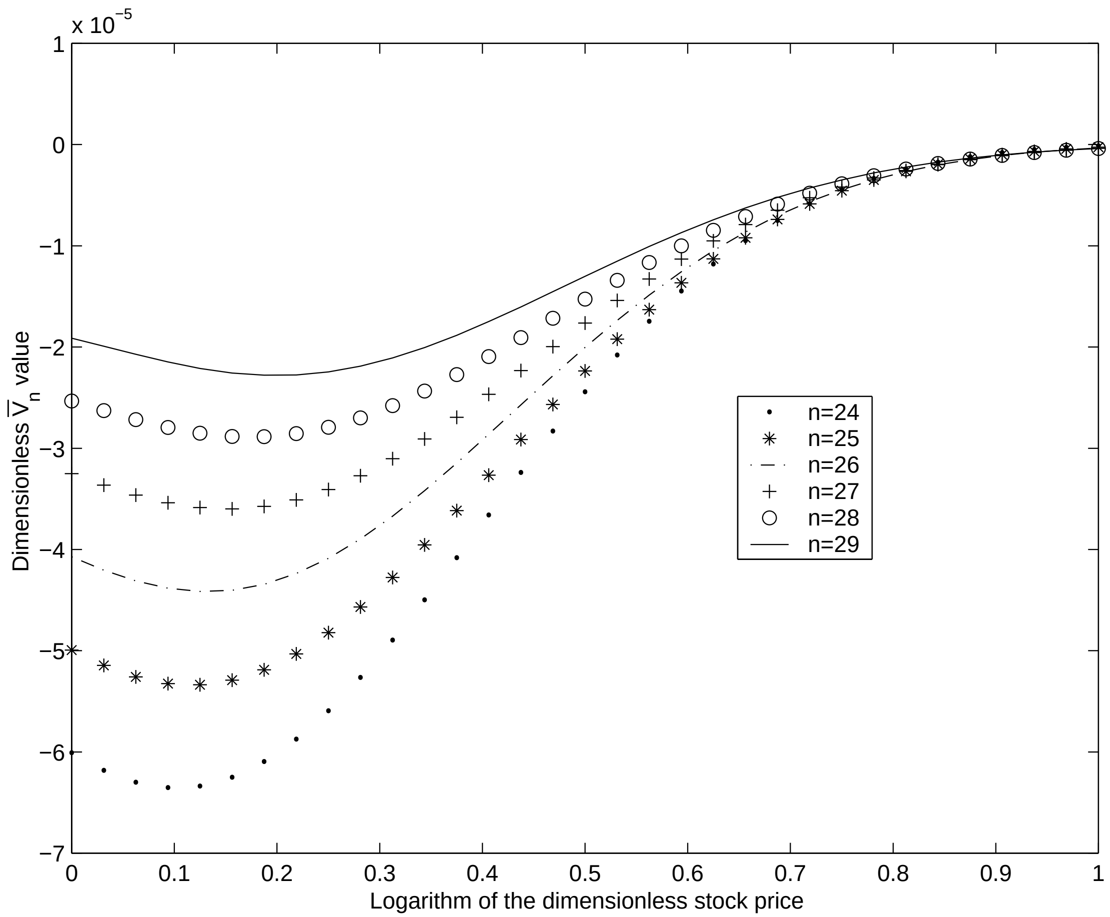

Figure 3: Dimensionless option prices with six different total summation numbers in Example 1

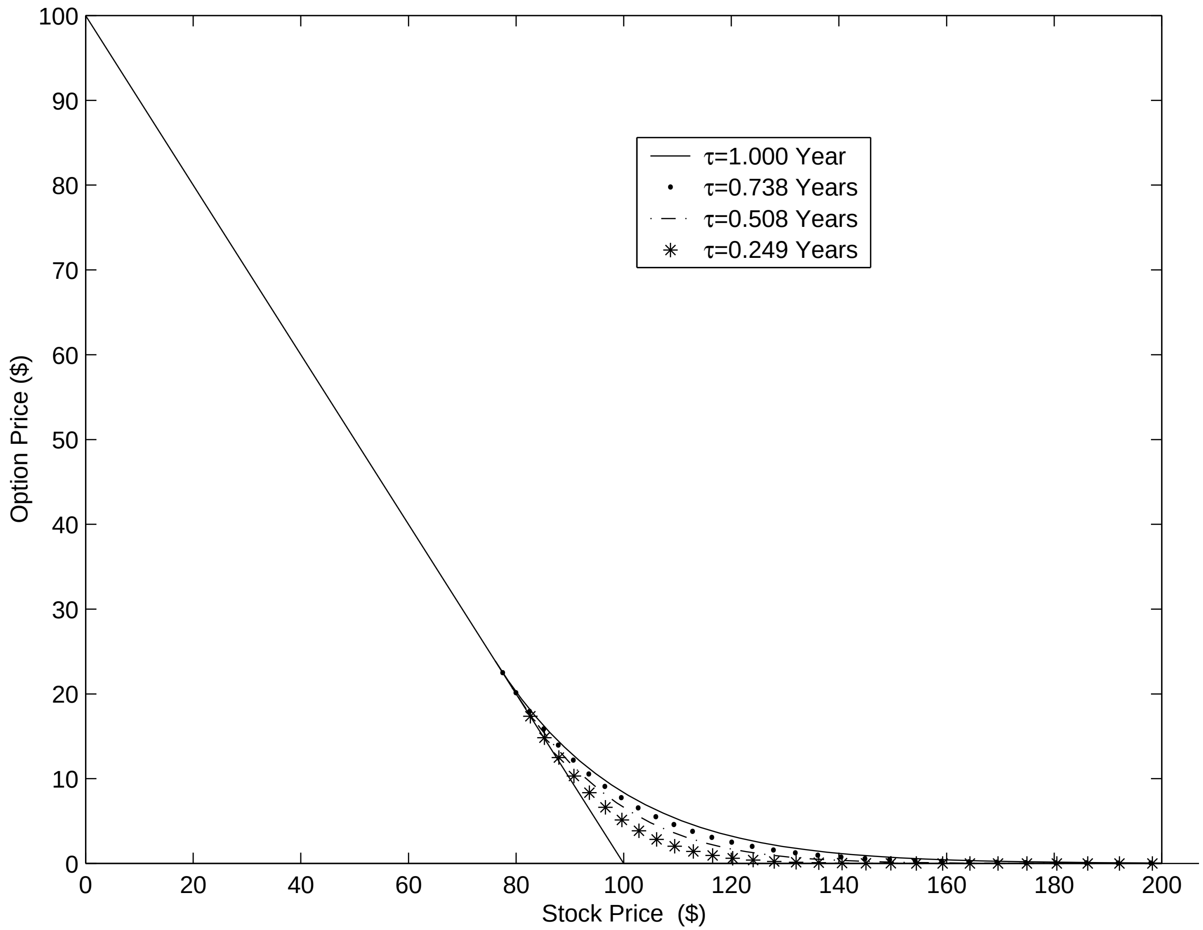

Figure 4: Option prices at different times to expiration in Example 1

Example 2:

Use the same example used in Bunch and Johnson (2000):

- **Strike price** $X = \$40$**,**
- **Risk-free interest rate** $r = 0.0488$**,**
- **Volatility** $\sigma = 0.3$**,**
- **Time to expiration** $T = 1$ **(year),**

In terms of the dimensionless variables, the two parameters involved are

- **Relative risk-free interest rate** $\gamma = 1.084$**,**
- **Dimensionless time to expiration** $\tau_{exp} = 0.045$**.**

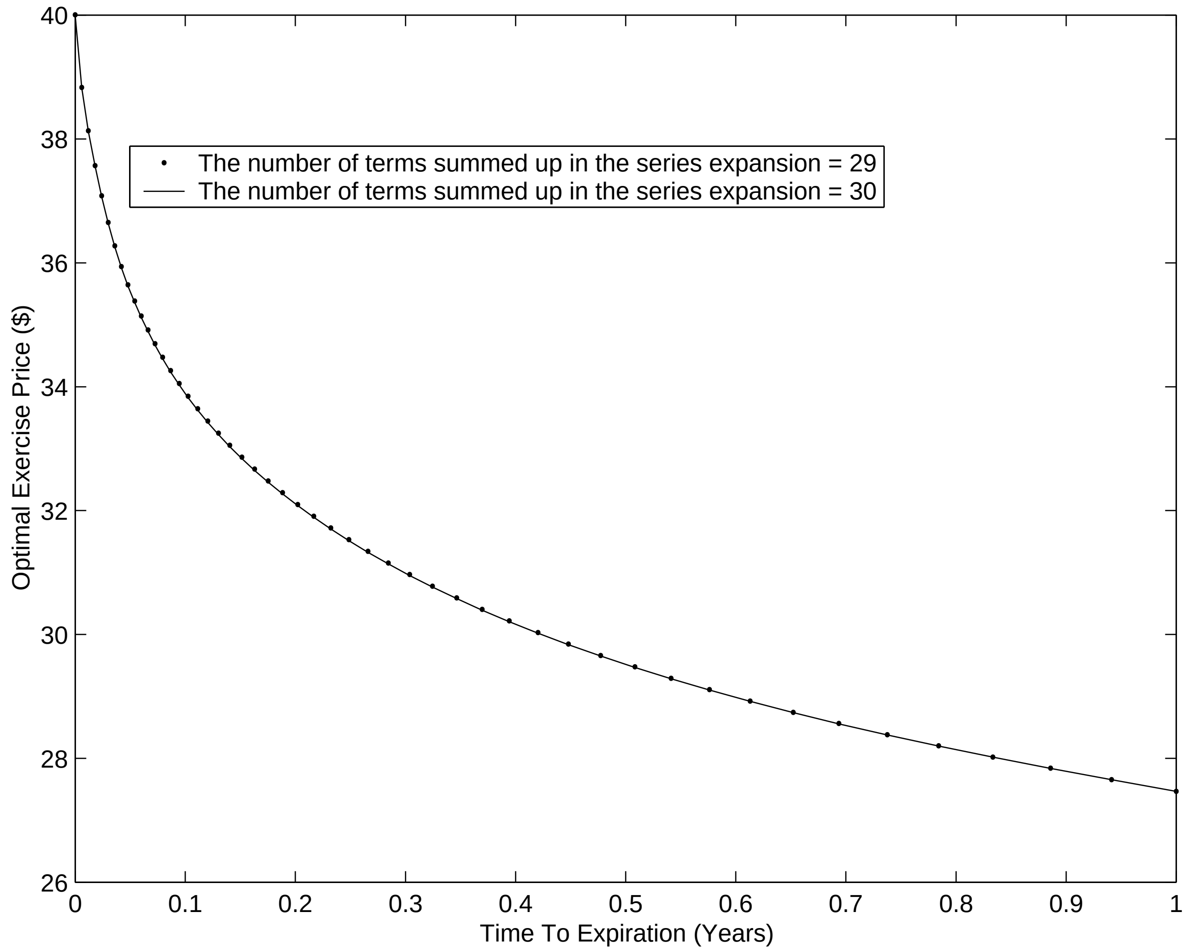

Figure 5: Convergence of the optimal exercise price in Example 2

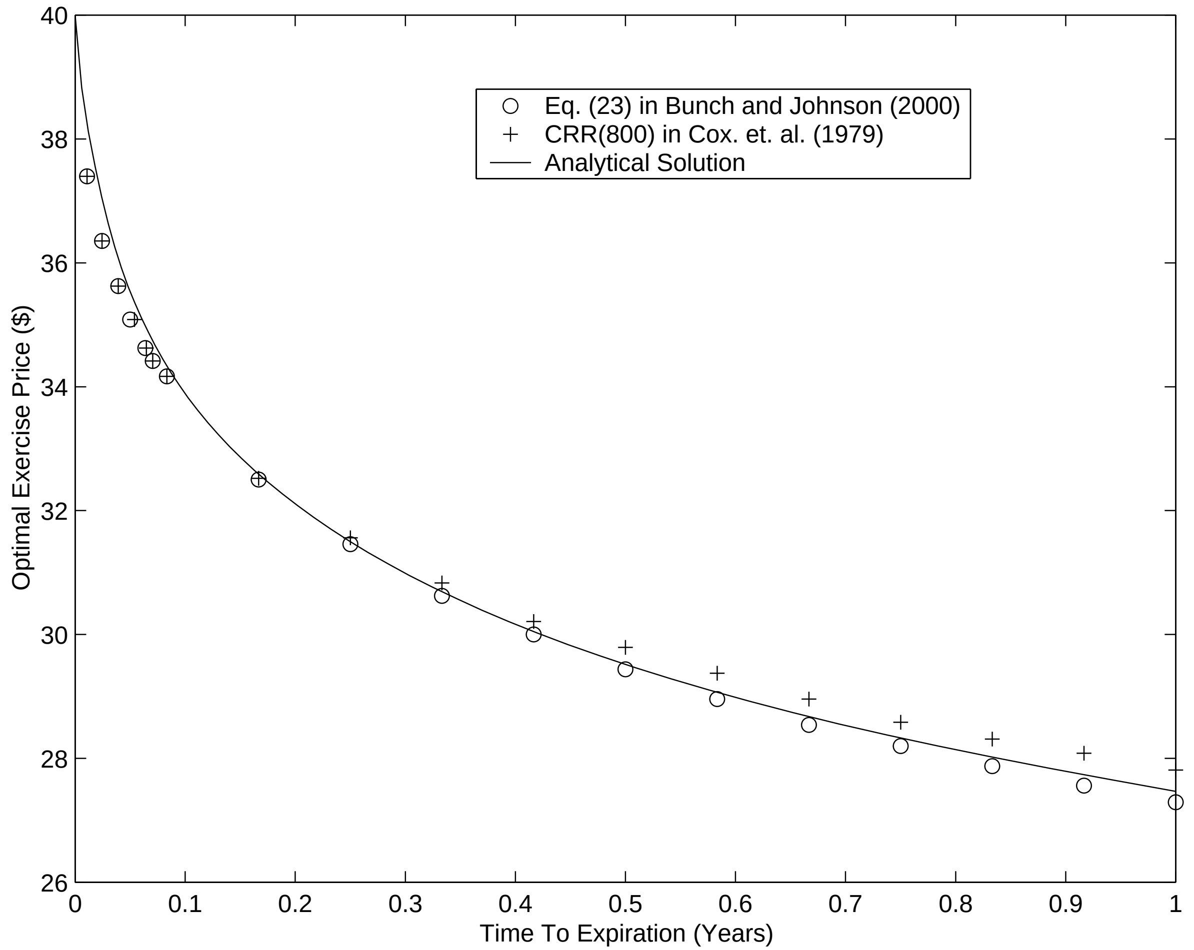

Figure 6: Optimal exercise prices for the case in Example 2

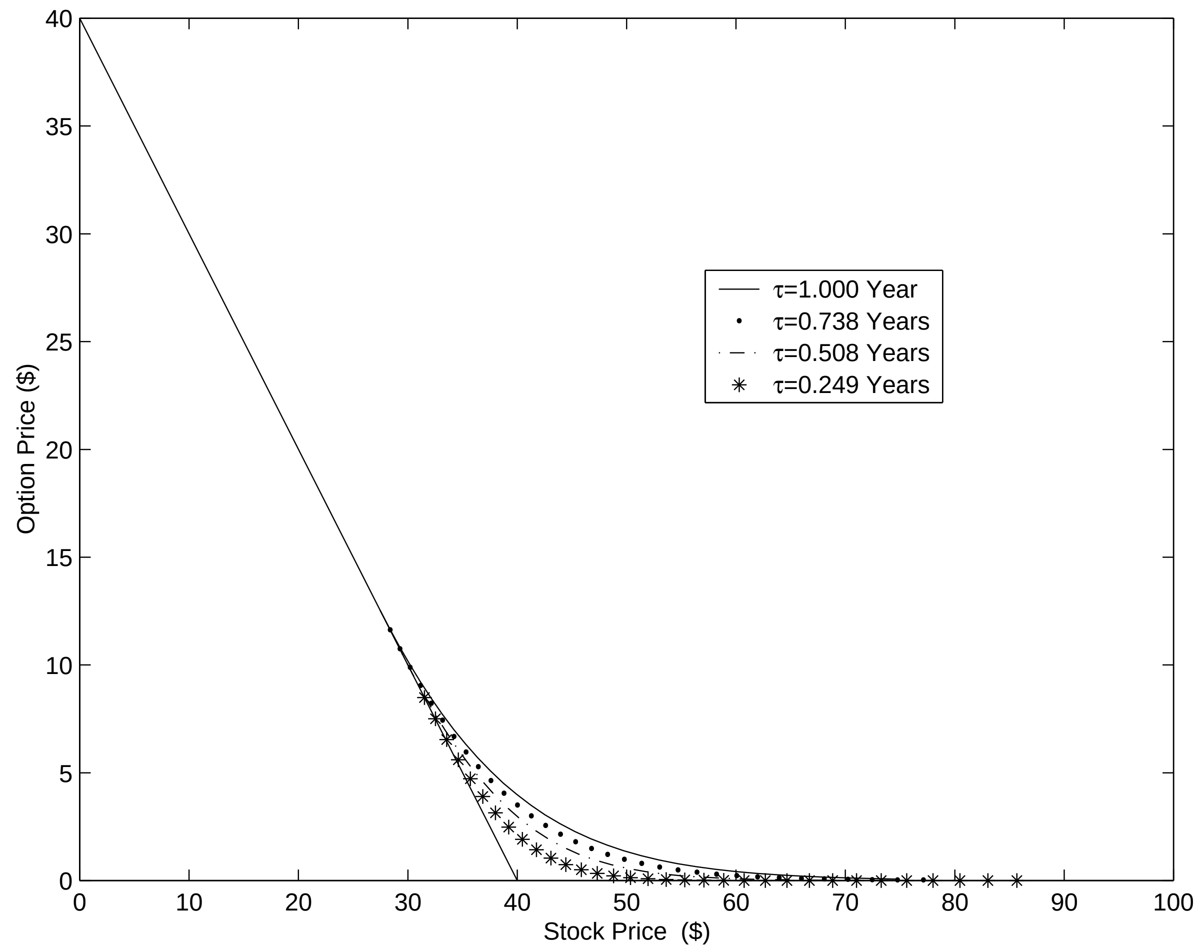

Figure 7: Option prices at different times to expiration in Example 2

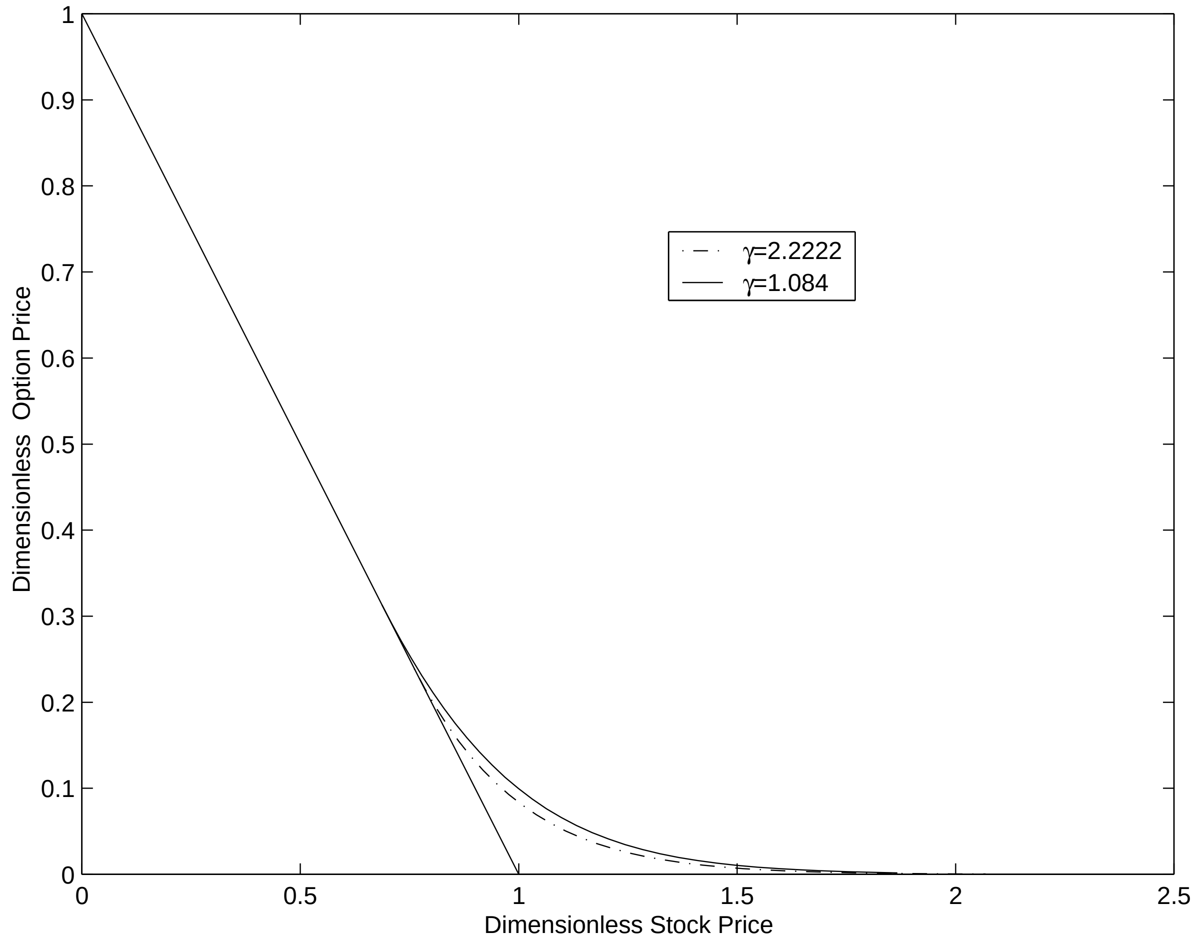

Figure 8: A comparison of dimensionless option price with different γ values

Example 3:

Use the same example used in Bunch and Johnson (2000):

- **Strike price** $X = \$100$,
- **Risk-free interest rate** $r = 0.01$,
- **Volatility** $\sigma = 0.2$,
- **Time to expiration** $T = 40$ **(years),**

In terms of the dimensionless variables, the two parameters involved are

- **Relative risk-free interest rate** $\gamma = 0.5$,
- **Dimensionless time to expiration** $\tau_{exp} = 0.4$.

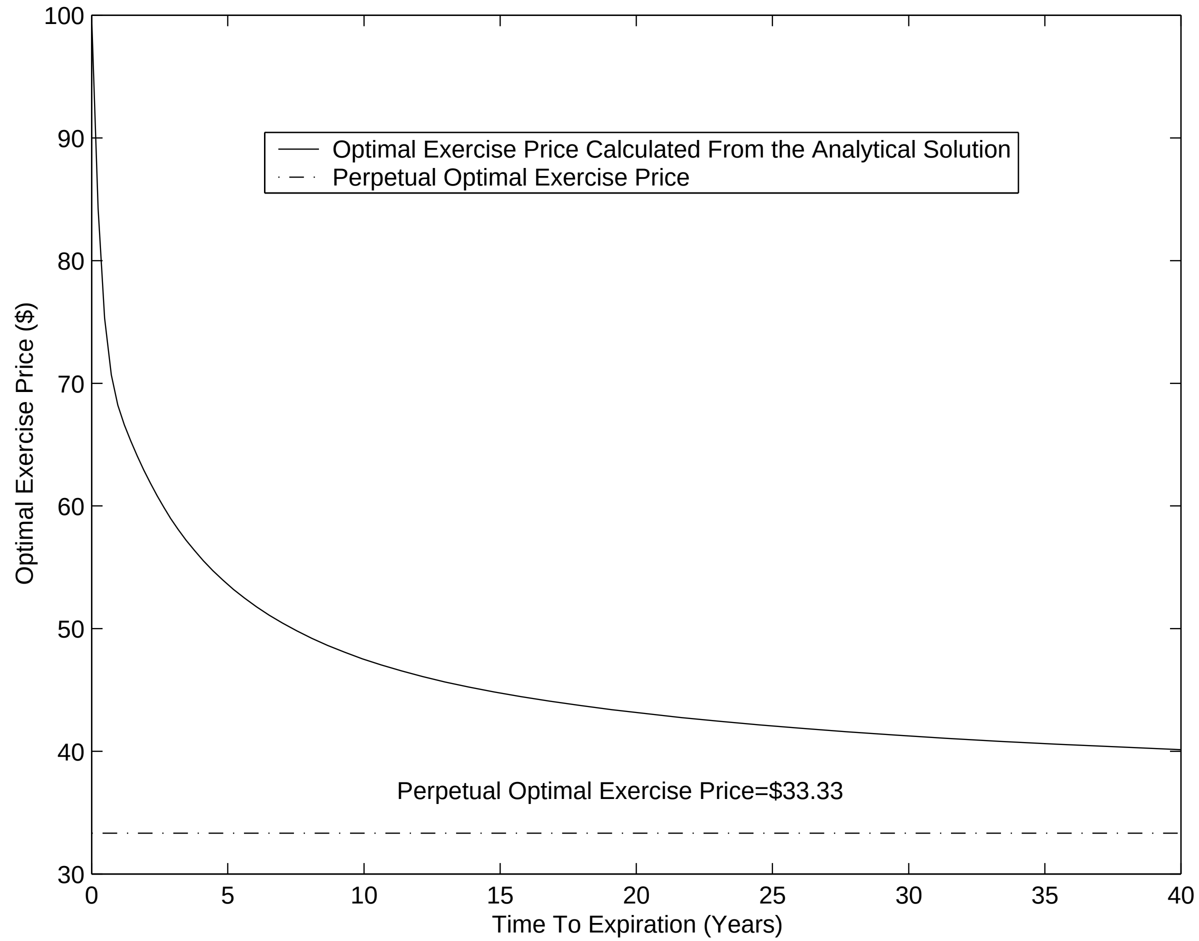

Figure 9: Optimal exercise price for the case in Example 3

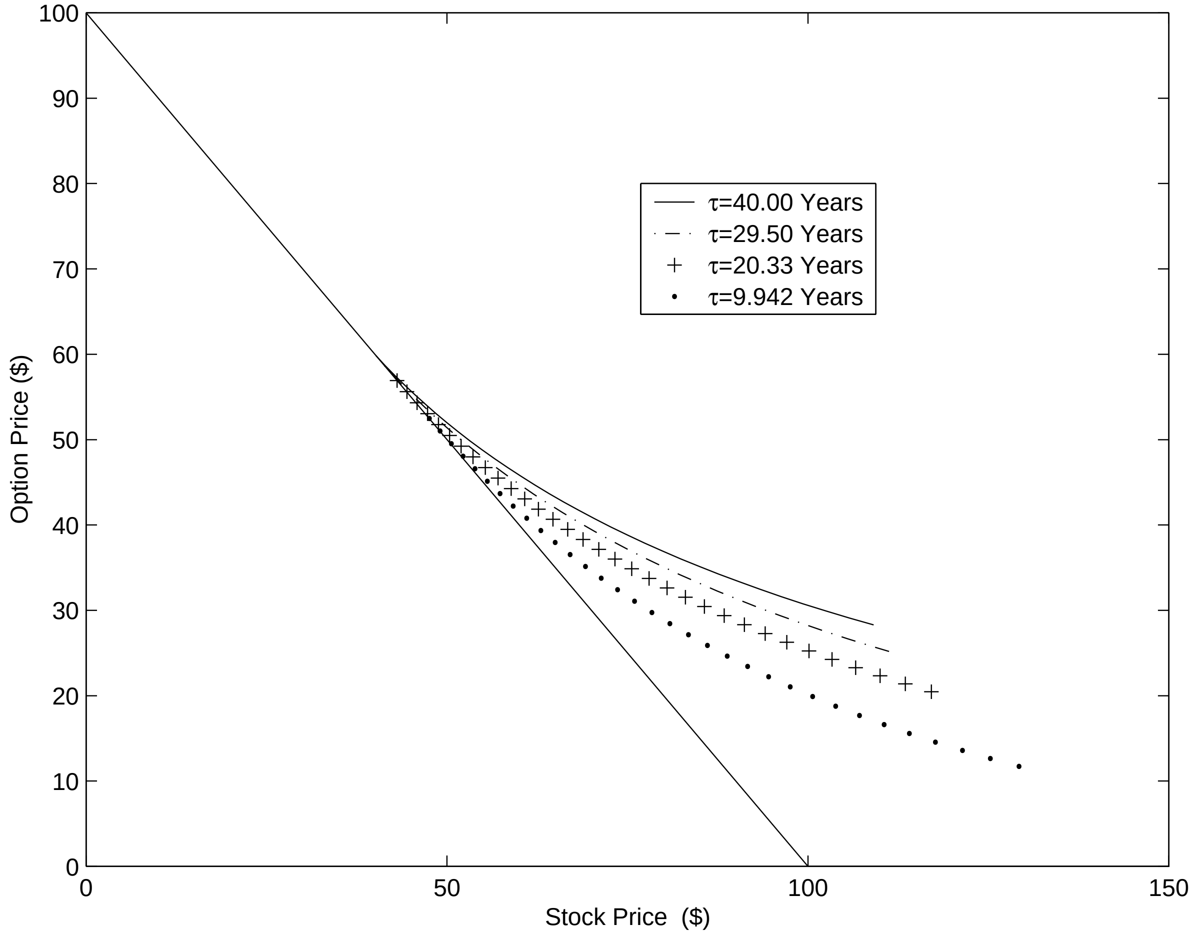

Figure 10: Option prices at different time to expiration in Example 3

Conclusions

- **With the homotopy-analysis method, a exact and explicit solution (in a Taylor series expansion form) of the well-known Black-Scholes equation is obtained for the first time;**
- **Numerical evidence has been provided to demonstrate the convergence of the series;**
- **The calculation of the Greeks can be easily carried out.**