Fourier Transform Techniques in Stochastic Volatility BGM

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Introduction

As an application of Fourier tranform analysis we show how to price a caplet in the framework of a stochastic volatility version of BGM.

First we specify the model dynamics which generalises shifted BGM by introducing a stochastic factor $\sqrt{V(t)}$ in the volatility term:

$$\frac{dK(t,T_j)}{K(t,T_j)+a(T_j)} = \sqrt{V(t)}\xi(t,T_j)dW_{j+1}(t),$$

where$K(t,T)$ is the time t value of the cash rate for maturity T, a is the maturity dependant shift, ξ a volatility term, and V follows the process

$$dV(t) = \lambda[\mu - V(t)]dt + \gamma\sqrt{V(t)}dU(t),$$

$with \quad \lambda, \mu, \gamma > 0, \quad V(0) = 1$, and with W and U independant Wiener processes. Thus V acts as a scaling term on the volatility.

We can use Fourier transform techniques to price a caplet in this framework by first transforming the model dynamics to make an affine system.

1 Background

(See Duffie Pan and Singleton: "Transform Analysis and Asset Pricing for Affine Jump Diffusions", and Filipovic: "Time Inhomogeneous Affine Processes".) Consider a Markov process satisfying an SDE of the form

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t,$$

where W_t is standard n-dimensional Brownian motion. The requirement that μ and $\sigma\sigma^T$ are both affine with respect to the state vector is equivalent to the characteristic function of the system being of exponential affine form. That is, the characteristic function of X_t conditional on X_s, $s < t$ can be written as

$$\mathbf{E}\{\exp(i\theta X_t)|\mathcal{F}_s\} = \exp(\phi(s,t,\theta) + \psi(s,t,\theta)^T X_s),$$

$$where \qquad \phi(s,t,\theta) \in \mathbb{C}\,, \quad \psi(s,t,\theta) \in \mathbb{C}^{\,n}.$$

We call such a process an Affine Process.

As an example of such a process we have the Heston stochastic volatility model, with correlation ρ, defined by:

$$dS_t = rS_t dt + \sqrt{V_t} S_t[\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}]$$

$$with \qquad dV_t = \lambda(\mu - V_t)dt + \gamma\sqrt{V_t} dW_t^{(1)}$$

2 Method

First I give a quick overview of the method of pricing a caplet in the stochastic volatility BGM setting:

- Starting with the model dynamics, we transform the model into an affine system
- Compute the characteristic function of the system at maturity, conditional on the state vector
- Take the Fourier transform of the caplet value, which we can find explicitly in terms of the characteristic function
- Take the inverse Fourier transform of this solution to find the price. This transform must be evaluated numerically

For a particular maturity T we specify the model dynamics

$$\frac{dK(t)}{K(t)+a} = \sqrt{V(t)}\xi(t)dW(t),$$

where

$$dV(t) = \lambda[\mu - V(t)]dt + \gamma\sqrt{V(t)}dU(t)$$

$$\lambda, \mu, \gamma > 0 \qquad V(0) = 1.$$

This is equivalent to the shifted Heston stochastic volatility model with zero correlation (and with V rescaled accordingly to incorporate the ξ volatility term). The main motivation for using such a model is that it allows (for a suitably high value of γ) one to approximately fit an implied volatility *smile*, as well as a skew.

We wish to find the time t value of an option struck at κ and maturing at time T, denoted by:

$$\mathbf{H}(t,\kappa) = \mathbf{E}\{[K(T)-\kappa]^+|\mathcal{F}_t\}.$$

Where $\mathbf{H}$ is for Heston option. We begin by making the substitution $S(t) = \ln(K(t)+a)$ and $k = \ln(\kappa + a)$, then we can express the present value of the option as

$$\mathbf{H}(k) = \mathbf{E}\{[K(T)-\kappa]^+|X(0)\} = \int_k^\infty (\exp s - \exp k)p(s)ds,$$

where $p(s)$ is the probability density function of $S(T)$ given $X(0)$.
The model dynamics become (by Ito's lemma)

$$dS(t) = -\frac{1}{2}V(t)\xi^2(t)dt + \sqrt{V(t)}\xi(t)dW(t),$$

$$dV(t) = \lambda[\mu - V(t)]dt + \gamma\sqrt{V(t)}dU(t),$$

$$V(0) = 1, \qquad S(0) = \ln(K(0)+a),$$

which is Markov with respect to the state vector $X(t) = (S(t), V(t))^T$ and affine in the sense that the drifts and the square of the volatilities are affine with respect to the state vector.

We can compute the above integral, in terms of the characteristic function of $S(T)$ conditional on the state vector $X(t)$, using fourier transform techniques (The following solution is courtesy of Alan Brace). First we must evaluate the characteristic function

$$\hat{p}(\theta;t) = \mathbf{E}\{\exp(i\theta S(T))|X(t)\}.$$

In order to ensure that the characteristic function has zero drift it must satisfy the PDE

$$\hat{p}_t - \frac{1}{2}\xi^2(t)V(t)\hat{p}_s + \lambda[\mu - V(t)]\hat{p}_v + \frac{1}{2}\xi^2(t)V(t)\hat{p}_{ss} + \frac{1}{2}\gamma^2 V(t)\hat{p}_{vv} = 0.$$

Where $\hat{p}_t = \frac{\partial\hat{p}(\theta;t)}{\partial t}$ etc. Also, as the system is affine in the aforementioned sense, the log of the characteristic function must be affine with respect to the state vector, i.e. it is of the form

$$\hat{p}(\theta;t) = \exp(A(t,T,\theta) + B(t,T,\theta)S(t) + C(t,T,\theta)V(t)),$$

where A, B and C are complex valued functons.
(Note: Since we are only concerned with a particular maturity we shall omit the variable T in the above functions A, B, C from here on. Also for a given θ we shall write A, B, and C as functions of t only if their is no risk of confusion.)

Then since $\hat{p}(\theta;T) = \exp(i\theta S(T))$, we must have $A(T) = C(T) = 0$, $B(T) = i\theta$.

Substituting this expression for $\hat{p}$ into the PDE yields

$$B_t S(t) + [C_t - \lambda C + \frac{1}{2}\gamma^2 C^2 + \frac{1}{2}\xi^2(t)(B^2 - B)]V(t) + A_t + \lambda\mu C = 0,$$

which implies

$$C_t - \lambda C + \frac{1}{2}\gamma^2 C^2 \frac{1}{2}(B^2 - B)\xi^2(t), \quad A_t + \lambda\mu C = 0, \quad B_t = 0.$$

So we see that $B(t) = i\theta$, and we make the substitution $C = (-\frac{1}{2}\gamma^2)^{-1}D$ to obtain

$$\begin{aligned} D_t &= D^2 + \lambda D - \frac{1}{4}\gamma^2(i\theta + \theta^2)\xi^2(t), \\ A_t &= \frac{2}{\gamma^2}\lambda\mu D(t), \\ A(T) &= D(T) = 0. \end{aligned}$$

This system has an analytic solution for $\xi(t)$ constant, so we make the assumtion that $\xi(t) = \xi_j$ is piecewise constant on each interval $(t_j, t_{j+1}]$ (where the time interval can be made as small as desired so as to approximate the function $\xi(t)$) and then using the boundary value $D(T) = A(T) = 0$ we can find recursively the functions $D(t)$ and $A(t)$ for each time interval. First we introduce the following notation:

$$\begin{aligned} \Delta_j &= \lambda^2 + \gamma^2(i\theta + \theta^2)\xi_j^2, \\ d_j^+ = \frac{1}{2}(-\lambda + \sqrt{\Delta_j}), &\quad d_j^- = \frac{1}{2}(-\lambda - \sqrt{\Delta_j}). \end{aligned}$$

The method of solution depends on the value of Δ:

Case $\Delta \neq 0$

First note

$$\begin{aligned} \frac{\sqrt{\Delta_j}}{D_s} &= \frac{\sqrt{\Delta_j}}{D^2 + \lambda D - \frac{1}{4}\gamma^2(i\theta + \theta^2)\xi^2} \\ &= \frac{D - d_j^- - D + d_j^+}{D^2 - (d_j^- + d_j^+)D + d_j^+ d_j^-} \\ &= \frac{1}{D - d_j^+} - \frac{1}{D - d_j^-} \end{aligned}$$

Then for $t \in (t_j, t_{j+1}]$,

$$
\begin{aligned}
\sqrt{\Delta_j}(t_{j+1} - t) &= \int_t^{t_{j+1}} \sqrt{\Delta_j} ds \\
&= \int_t^{t_{j+1}} \sqrt{\Delta_j} \frac{dD}{D_s} \\
&= \int_t^{t_{j+1}} \frac{dD}{D - d_j^+} - \int_t^{t_{j+1}} \frac{dD}{D - d_j^-} \\
&= \ln(\frac{D - d_j^-}{D - d_j^+} \frac{D_{j+1} - d_j^+}{D_{j+1} - d_j^-}) \\
\textit{giving} \qquad D(t) &= \frac{d_j^+ - d_j^- [\frac{D_{j+1} - d_j^+}{D_{j+1} - d_j^-} \exp(-\sqrt{\Delta_j}(t_{j+1} - t))]}{1 - \frac{D_{j+1} - d_j^+}{D_{j+1} - d_j^-} \exp(-\sqrt{\Delta_j}(t_{j+1} - t))}, \\
\textit{then} \qquad A(t) &= A_{j+1} - \int_t^{t_{j+1}} \frac{2}{\gamma^2} \lambda \mu D ds \\
&= A_{j+1} - \frac{2}{\gamma^2} \lambda \mu \frac{1}{\sqrt{\Delta_j}} \int_t^{t_{j+1}} \frac{d_j^+ - d_j^- y}{(1 - y) y} dy \\
&= A_{j+1} - \frac{2}{\gamma^2} \lambda \mu [\frac{d_j^+}{\sqrt{\Delta_j}} \ln \frac{y(t_{j+1})}{y(t)} - \ln \frac{y(t_{j+1}) - 1}{y(t) - 1}], \\
\textit{where} \qquad y(t) &= \frac{D_{j+1} - d_j^+}{D_{j+1} - d_j^-} \exp(-\sqrt{\Delta_j}(t_{j+1} - t)).
\end{aligned}
$$

For large time to maturity and reasonably high values of γ (around 1), as we recursively find the values $D(t)$ and $A(t)$ for earlier intervals, we find that $D_{j+1} - d_j^+ \longrightarrow 0$ leading to numerical problems with a zero in the denominator. So instead substitute in $y(t)$ to the expression for $A(t)$ giving

$$
A(t) = A_{j+1} - \frac{2}{\gamma^2} \lambda \mu \{ d_j^+ (t_{j+1} - t) - \ln(\frac{y(t_{j+1}) - 1}{y(t) - 1}) \}.
$$

Case $\Delta = 0$

In this case we have, for $t \in (t_j, t_{j+1}]$,

$$
\begin{aligned}
(t_{j+1}-t) &= \int_t^{t_{j+1}} ds = \int_t^{t_{j+1}} \frac{dD}{D_t} \\
&= \int_t^{t_{j+1}} \frac{dD}{D^2+\lambda D+\frac{1}{4}\lambda^2} \\
&= \int_t^{t_{j+1}} \frac{dD}{(D-d_j^+)^2} \\
&= \frac{1}{D-d_j^+} - \frac{1}{D_{j+1}-d_j^+},
\end{aligned}
$$

$$
\begin{aligned}
\textit{giving} \qquad D(t) &= \frac{D_{j+1}+d_j^+ y(t)}{1+y(t)}, \\
\textit{where} \qquad y(t) &= (D_{j+1}-d_j^+)(t_{j+1}-t), \\
\textit{and so} \qquad A(t) &= A_{j+1} + \frac{2}{\gamma^2}\lambda\mu \frac{1}{(D_{j+1}-d_j^+)} \int_t^{t_{j+1}} \frac{D_{j+1}+d_j^+ y}{1+y} dy \\
&= A_{j+1} - \frac{2}{\gamma^2}\lambda\mu\{d_j^+(t_{j+1}-t) + \ln[1+(D_{j+1}-d_j^+)(t_{j+1}-t)]\}.
\end{aligned}
$$

So we can find D and A recursively from $t=T$ to $t=0$ by using the fact that $D(T)=A(T)=0$ and letting the solution for the beginning of each interval as the boundary value for the next iteration. We can then express the characteristic function of $S(T)$ at $t=0$ (which it turns out is what we need) as

$$
\begin{aligned}
\hat{p}(\theta;t) &= \exp(A(t) - (\frac{1}{2}\gamma^2)^{-1} D(t)V(t) + i\theta S(t)), \\
\textit{so} \qquad \hat{p}(\theta;0) &= \exp(A(0) - (\frac{1}{2}\gamma^2)^{-1} D(0) + i\theta S(0)) \qquad (\textit{recall}\, V(0)=1).
\end{aligned}
$$

We are now in a position to evaluate the price of an option. Recall

$$\mathbf{H}(k) = \int_k^\infty (\exp s - \exp k) p(s) ds.$$

Define, for real $b>0$,

$$\mathbf{H}_b(k) = \exp(bk)\mathbf{H}(\mathbf{k}).$$

Thus we have $\lim_{|k|\to\infty} \mathbf{H}_b(k) = 0$. The Fourier transform of $\mathbf{H}_b$ is then given by

$$\begin{aligned}
\hat{\mathbf{H}}_b(x) &= \int_{-\infty}^{\infty} \exp(bk)\mathbf{H}(k)\exp(2\pi ixk)dk \\
&= \int_{-\infty}^{\infty} \exp(bk)\{\int_k^{\infty} p(s)(\exp s - \exp k)ds\}\exp(2\pi ixk)dk \\
&= \int_{-\infty}^{\infty} \{p(s)\int_{-\infty}^{s} [\exp(s+bk+2\pi ixk) - \exp(k+bk+2\pi ixk)]dk\}ds \\
&= \int_{-\infty}^{\infty} p(s)\exp(s+bs+2\pi ixs)(\frac{1}{b+2\pi ix} - \frac{1}{1+b+2\pi ix})ds \\
&= \frac{\hat{p}(2\pi x - i(1+b), 0)}{b+b^2-4\pi^2x^2+2\pi ix(2b+1)},
\end{aligned}$$

and so on taking the inverse Fourier transform we obtain

$$\begin{aligned}
\mathbf{H}(k) &= \exp(-bk)\int_{-\infty}^{\infty} \hat{\mathbf{H}}_b(x)\exp(-2\pi ikx)dx \\
&= 2\exp(-bk)\int_0^{\infty} \mathrm{Re}\{\hat{\mathbf{H}}_b(x)\exp(-2\pi ikx)\}dx.
\end{aligned}$$

The last step follows because $\hat{\mathbf{H}}_b(-x) = \overline{\hat{\mathbf{H}}_b(x)}$. Note that if $b > 0$ then there is no singularity in the integrand. We have found $b = \frac{1}{2}$ to be a good default value.

We can easily use the above methodology to value a caplet. We have the time t value of a caplet, struck at κ, maturing at T, and paying at T_1, is given by

$$\mathrm{Cpl}(t) = \delta B(t, T_1)\mathbf{E}_{T_1}\{[K(T,T) - \kappa]^+|\mathcal{F}_t\},$$

where $\delta = T_1 - T$, and $B(t,T)$ is the time t value of a T maturing zero coupon bond.

3 Results

The above methodology was implemented using Matlab. I briefly summarise the method followed in implementation:

- The problem boils down to numerically integrating the inverse Fourier transform
- For each x value on $[0,\infty)$, the integrand must be calculated recursively
- The x value defines a value of θ by $\theta = 2\pi x - i(1+b)$ which must then be fed into the method detailed above for calculating the characteristic function of $S(T)$ for a given value of θ

The program was tested to ensure that put-call parity was satisfied and also that the Balck caplet values were returned when the volatility was made approximately deterministic and the shift set to zero. The graphs below illustrate this.

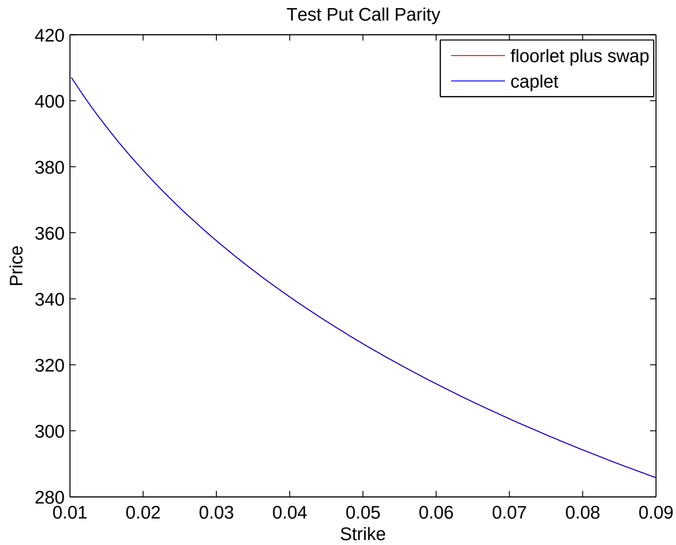
The parameters used were $\xi \equiv 50\%$, $a = 0$, $\gamma = 2$, $\mu = \lambda = 1$, $T = 20$, on a face value of \$100 000. For testing the model method against the Black caplet values the same parameters were used, except $\gamma = 0.001$.

The Pricing routine could then be used to calibrate the a and ξ parameters in the model to market caplet volatility data. As stated above, this model allows one to approximately fit a smile, which is what is observed in the market for the short maturaties. For this implementation the stochastic parameters were chosen by eye to approximately fit the curvature of the smiles away from the money, and the volatility (ξ) and shift parameters were used to fit the value and slope of the volatility (respectively) at the money. This was achieved by the non linear least squares method in Matlab. The stochastic parameters used were $\mu = 1$, $\gamma = 1.9$, and $\lambda = \frac{3\gamma^2}{4\mu}$ (this relationship is necessary for simulation in this model). The graphs below show the implied volatilities generated by the stochastic volatility BGM model, compared with the market implied volatilities, for a range of maturities ($T = 0.25, 0.5, 1, 2, 3, 5, 7, 10, 12, 15, 19.75$ years). For comparison we show graphs of the impled volatilities generated by a shifted non-stochastic volatility BGM for the maturities $T = 0.5, 1, 15$. It can be seen that the stochastic volatility approximates the curvature of the smile more accurately, particularly for the shorter maturities.

4 Further Work

The work carried out thus far leaves several things left to be explored:

- The integral taking the Fourier inversion of $\mathbf{H}_b$ was approximated using a quadrature rule (`quadl`, inbuilt into Matlab). It is likely that another method may be faster, however work needs to be done to determine the accuracy and speed of other methods of approximating this integral.
- In calculating the functions A, and D of the characteristic function it was assumed that they were continuous in t. This holds if and only if the affine system described is stochastically continuous. That is if the transition function $p_{s,T}(x,.)$ converges weakly to $p_{t,T}(x,.)$ on the state space as $s \longrightarrow t$. This was not proved.
- A robust method for calibrating the parameters λ, μ and γ to market data needs to be implemented.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floorlet plus swap
caplet
Price
Strike
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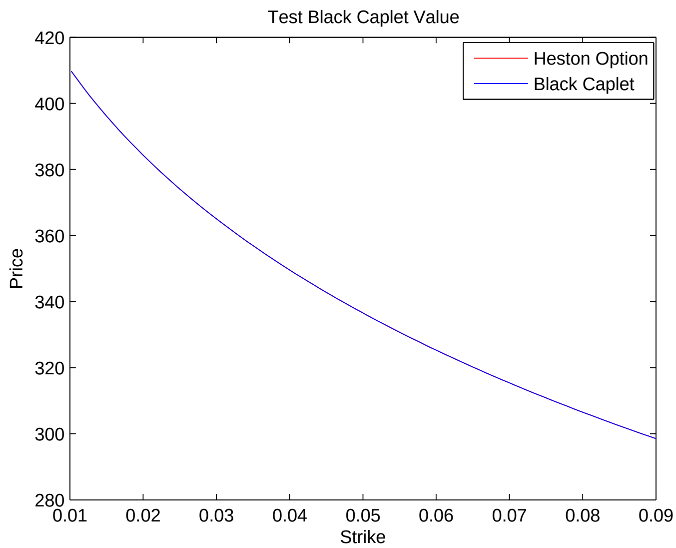
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Black Caplet
Price
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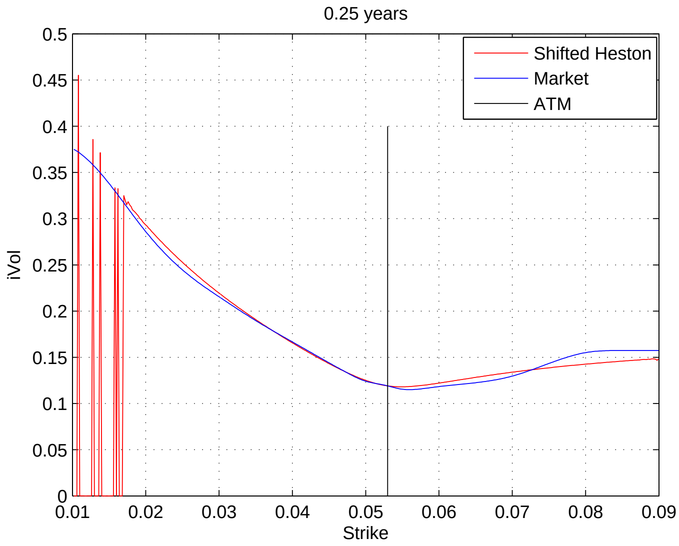
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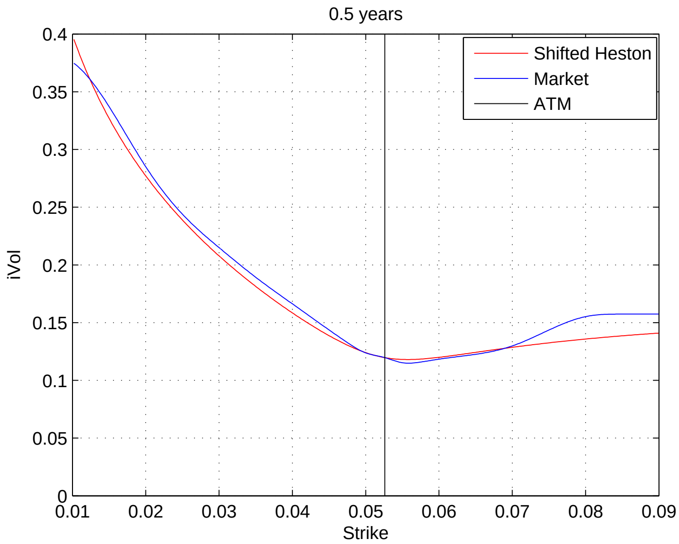
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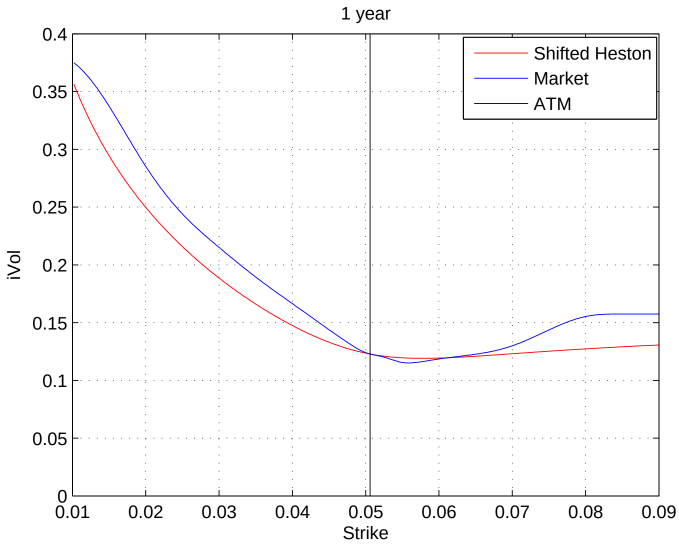
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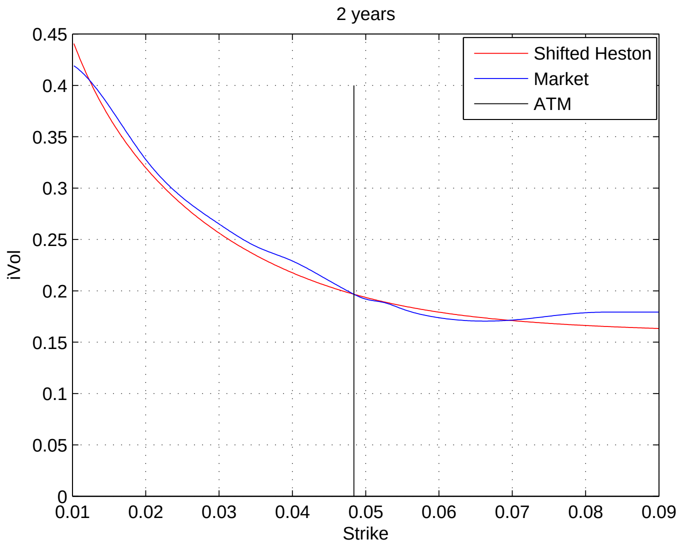
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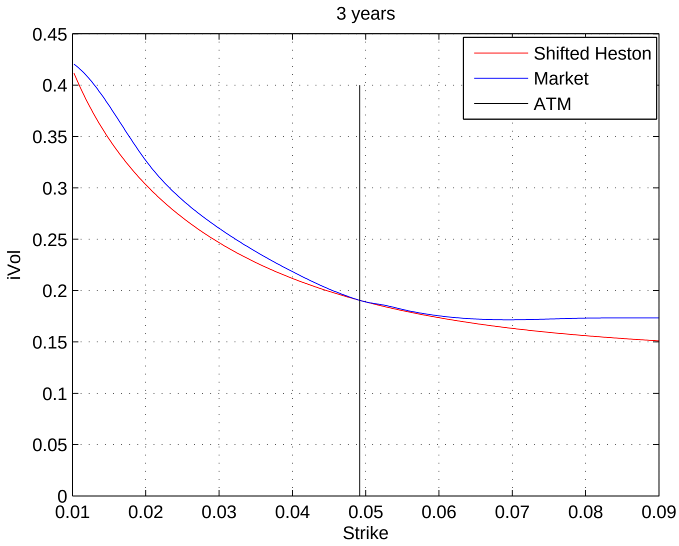
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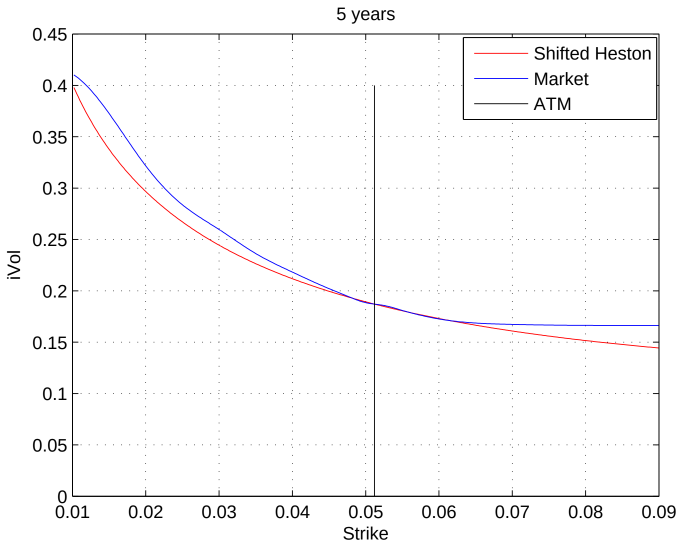
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ATM
iVol
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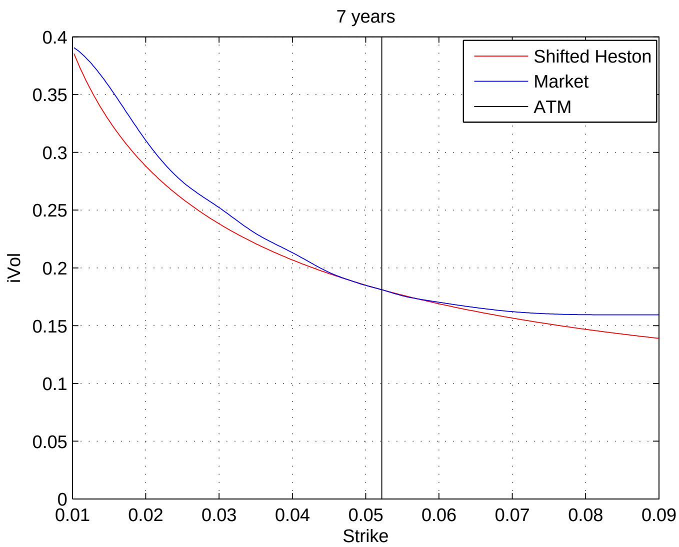
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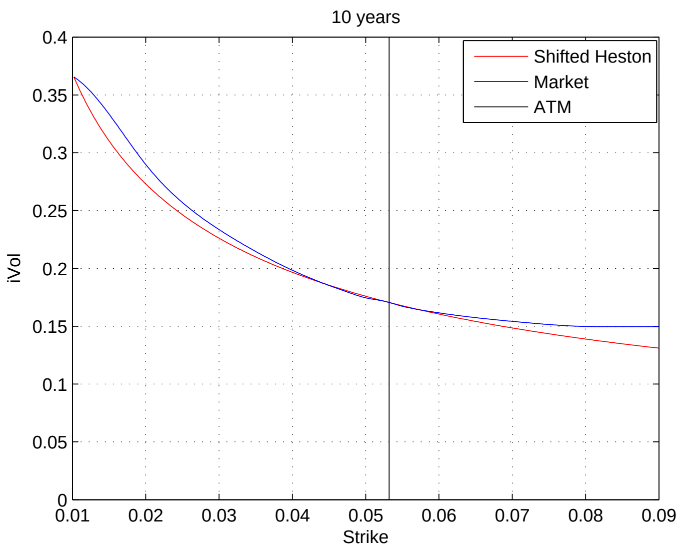
ATM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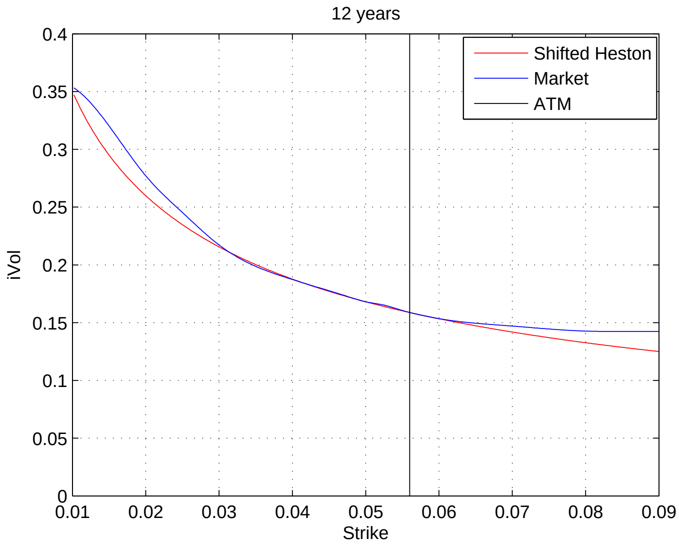
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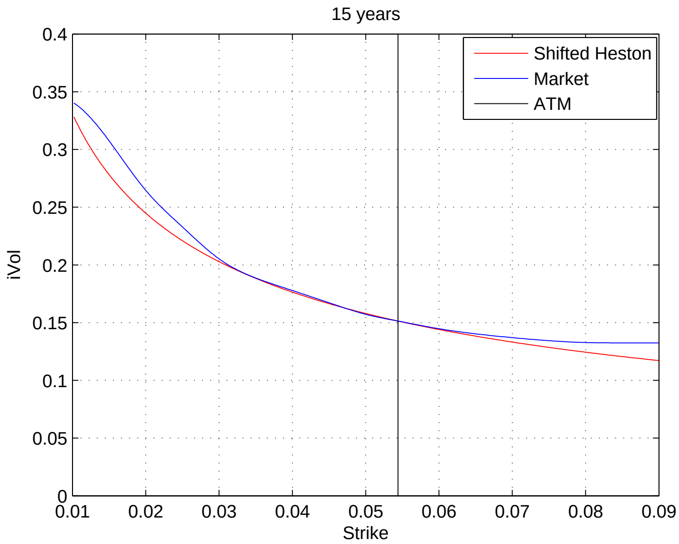
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iVol
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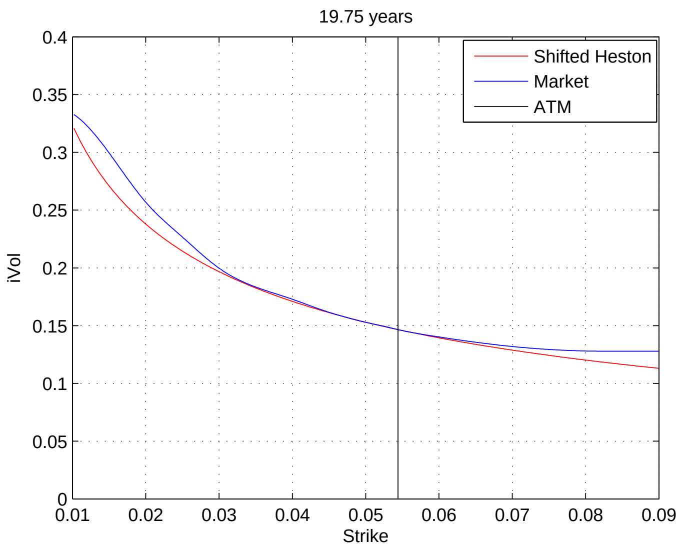
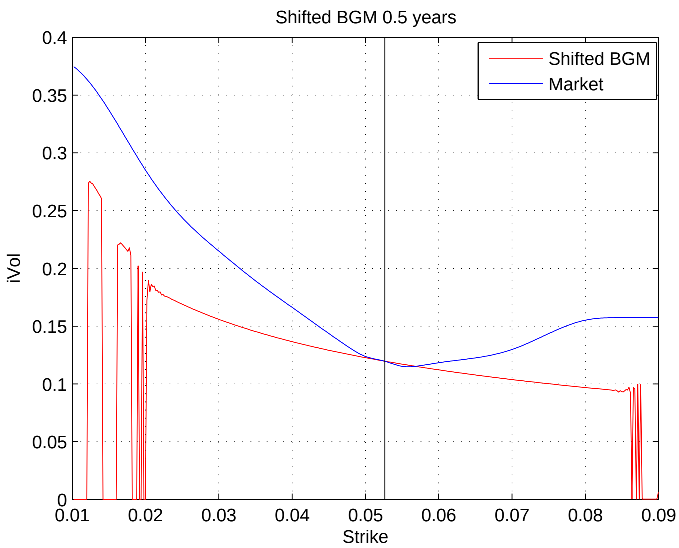

19.75 years
Shifted Heston
Market
ATM
iVol
Strike


Shifted BGM 0.5 years
Shifted BGM
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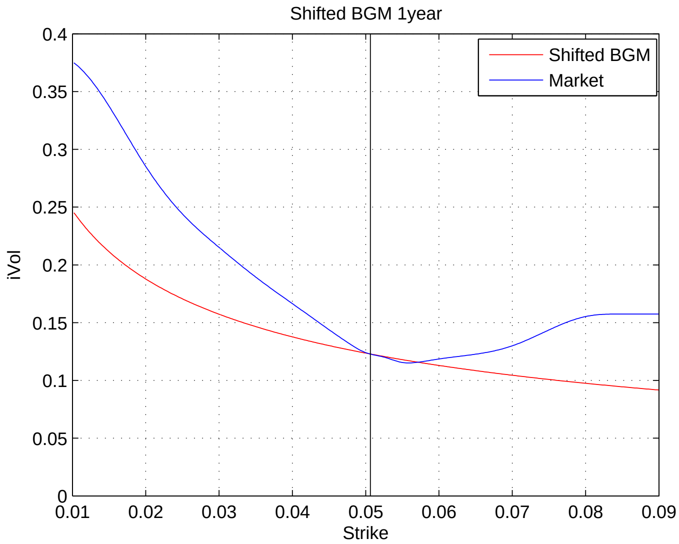
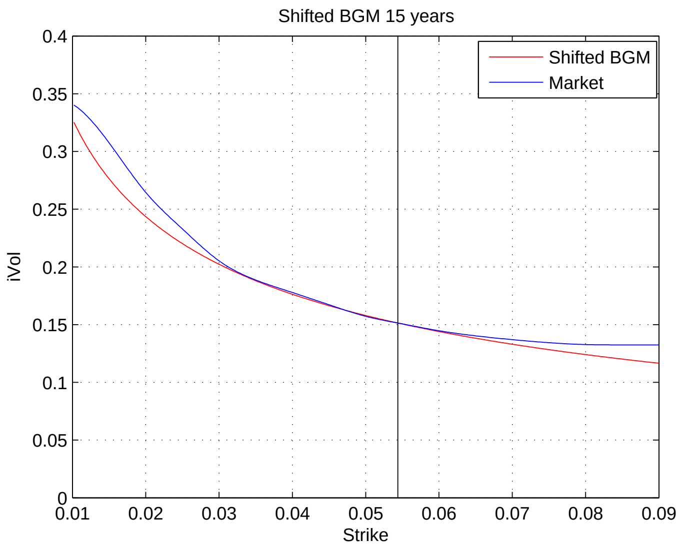
iVol
Strike


Shifted BGM 1year
Shifted BGM
Market
iVol
Strike
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09


Shifted BGM 15 years
Shifted BGM
Market
iVol
Strike
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09

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