Fourier Space Time-stepping Framework for Option Pricing with Lévy Models

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Overview

Develop a framework for numerical pricing of financial derivatives that

- Precise, fast and rapidly convergent
- Applicable to pricing of a wide range of European and path-dependent, single- and multi-asset, vanilla and exotic derivatives
- Efficiently handles path-independent and discretely monitored derivatives
- Generically handles various spot-price models and option payoffs

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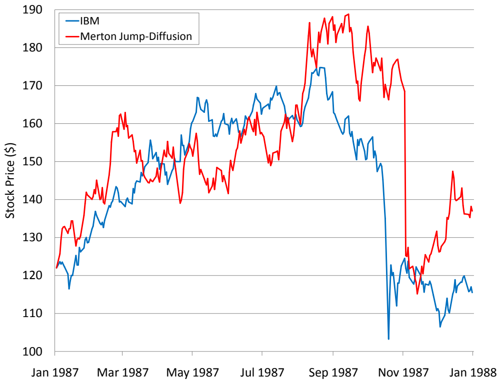
The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space and solve the ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

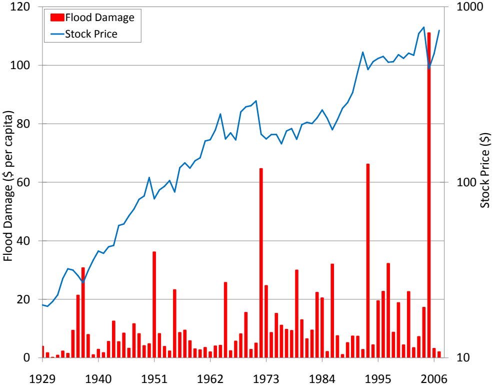
Equity Derivatives



Independent-increment exponential Lévy models

- Geometric Brownian motion
- Merton/Kou jump-diffusion
- Exponential Lévy models (VG, NIG, CGMY)

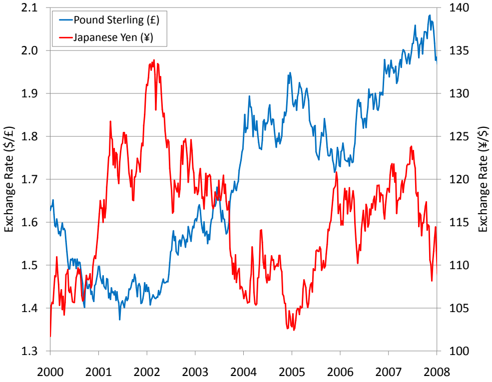
Insurance Derivatives



Joint evolution of insurance losses and asset prices model

- Losses are random, driven by Poisson process
- Jumps in asset price is driven by loss arrival

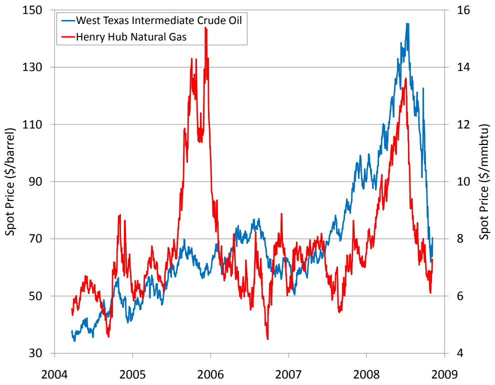
Currency Derivatives



Regime-switching exponential Lévy models

- Merton/Kou jump-diffusion with stochastic volatility
- Exponential Lévy models with stochastic volatility and skew

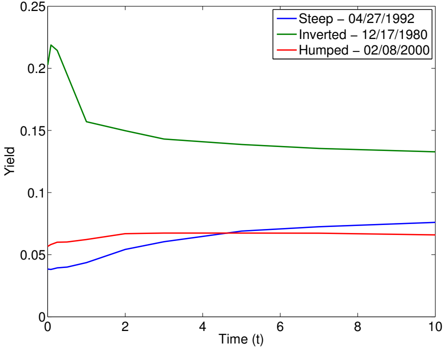
Commodity Derivatives



Mean-reverting jump-diffusion models

- Mean-reverting jump-diffusion with stochastic long-run level
- Decoupled mean-reverting diffusions and jumps
- Co-dependent jumps and correlated diffusions

Interest-rate Derivatives



Mean-reverting jump-diffusion short-rate models

- Time-inhomogeneous preference-free models:
  Vasicek++ model (a.k.a. extended Vasicek or Hull-White model)
- Two-factor time-inhomogeneous preference-free models:
  Hull-White 2-factor model and the equivalent G2++ model
- Jump extended Vasicek++ and G2++ models

Numerical Methods for Option Pricing

- Monte Carlo methods
- Tree methods
- Finite difference methods
  - Alternating Direction Implicit-FFT - Andersen and Andreasen (2000)
  - Implicit-Explicit (IMEX) - Cont and Tankov (2004)
  - IMEX Runge-Kutta - Briani, Natalini, and Russo (2004)
  - Fixed Point Iteration - d'Halluin, Forsyth, and Vetzal (2005)
- Quadrature methods
  - Reiner (2001)
  - QUAD - Andricopoulos, Widdicks, Duck, and Newton (2003)
  - Q-FFT - O'Sullivan (2005)
- Transform-based methods
  - Carr and Madan (1999)
  - Raible (2000)
  - Lewis (2001)
  - Lord, Fang, Bervoets, and Oosterlee (2008)

1. Introduction
2. Fourier Space Time-stepping method
3. Regime-Switching FST method
4. Mean-Reverting FST method
5. Interest-Rate FST method
6. Conclusions

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Independent Increment Pricing Framework

The Model

$$\mathbf{S}(t) = \mathbf{S}(0)\, e^{\mathbf{X}(t)}$$

where $\mathbf{X}(t)$ is a Lévy process with characteristic triplet $(\gamma, \mathbf{\Sigma}, \boldsymbol{\nu})$

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where $\mathbf{X}(t)$ is a Lévy process with characteristic triplet $(\gamma, \mathbf{\Sigma}, \boldsymbol{\nu})$

- The discount-adjusted and log-transformed price process $v(t, \mathbf{X}(t)) \triangleq e^{r(T-t)} V(t, \mathbf{S}(0) e^{\mathbf{X}(t)})$ satisfies a PIDE

$$\begin{cases} (\partial_t + \mathcal{L})\, v(t, \mathbf{x}) & = \mathbf{0}\,, \\ v(T, \mathbf{x}) & = \varphi(\mathbf{S}(0)\, e^{\mathbf{x}})\,, \end{cases}$$

where $\mathcal{L}$ is the infinitesimal generator of the Lévy process:

$$\mathcal{L}g(\mathbf{x}) = \left(\gamma' \partial_{\mathbf{x}} + \tfrac{1}{2}\, \partial_{\mathbf{x}}' \mathbf{\Sigma} \partial_{\mathbf{x}}\right) g(\mathbf{x}) + \int \left(g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}) - \mathbb{1}_{\{|\mathbf{y}|<1\}} \mathbf{y}' \partial_{\mathbf{x}} g(\mathbf{x})\right) \nu(d\mathbf{y})$$

Fourier Transform

- A function in the space domain $g(\mathbf{x})$ can be transformed to a function in the frequency domain $\hat{g}(\boldsymbol{\omega})$, where $\boldsymbol{\omega}$ is given in radians per second, and vice-versa using the continuous Fourier transform

$$\mathcal{F}\left[g\right](\boldsymbol{\omega}) \triangleq \int_{-\infty}^{\infty} g(\mathbf{x}) e^{-i\boldsymbol{\omega}'\mathbf{x}} d\mathbf{x}$$

$$\mathcal{F}^{-1}\left[\hat{g}\right](\mathbf{x}) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\boldsymbol{\omega}) e^{i\boldsymbol{\omega}'\mathbf{x}} d\boldsymbol{\omega}$$

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- Continuous Fourier transform is a linear operator that maps spatial derivatives ∂_x into multiplications in the frequency domain

$$\mathcal{F}[\partial_{\mathbf{x}}^n g](\boldsymbol{\omega}) = i\boldsymbol{\omega}\, \mathcal{F}\left[\partial_{\mathbf{x}}^{n-1} g\right](\boldsymbol{\omega}) = \cdots = (i\boldsymbol{\omega})^n \mathcal{F}[g](\boldsymbol{\omega})$$

Pricing Framework in Fourier Space

- Applying the Fourier transform to the infinitesimal generator $\mathcal{L}$ of $\mathbf{X}(t)$ allows the characteristic exponent $\Psi(\omega)$ to be factored out:

$$\mathcal{F}\left[\mathcal{L}v\right](t,\omega) = \Psi(\omega)\mathcal{F}\left[v\right](t,\omega)\,,$$

where

$$\Psi(\omega) = i\gamma'\omega - \tfrac{1}{2}\,\omega'\mathbf{\Sigma}\omega + \int \left(e^{i\omega'\mathbf{y}} - 1 - i\mathbb{1}_{\{|\mathbf{y}|<1\}}\omega'\mathbf{y}\right)\nu(d\mathbf{y})$$

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- The PIDE is therefore transformed into a d-parameter family of ODEs parameterized by ω:

$$\begin{cases} \partial_t\mathcal{F}[v](t,\omega) + \Psi(\omega)\mathcal{F}[v](t,\omega) & = 0\,, \\ \mathcal{F}[v](T,\omega) & = \mathcal{F}[\varphi](\omega) \end{cases}$$

Characteristic Exponents for Lévy Processes

Model	Characteristic Exponent $\Psi(\omega)$
Black-Scholes-Merton	$i\gamma\omega - \frac{\sigma^2\omega^2}{2}$
Merton jump-diffusion	$i\gamma\omega - \frac{\sigma^2\omega^2}{2} + \lambda(e^{i\tilde{\mu}\omega - \tilde{\sigma}^2\omega^2/2} - 1)$
Kou jump-diffusion	$i\gamma\omega - \frac{\sigma^2\omega^2}{2} + \lambda(\frac{\eta_p}{1-i\omega\eta_+} + \frac{1-\eta_p}{1+i\omega\eta_-} - 1)$
Variance Gamma	$-\frac{1}{\mu}\ln(1 - i\gamma\mu\omega + \frac{\sigma^2\mu\omega^2}{2})$
Normal Inverse Gaussian	$-\frac{1}{\mu}(\sqrt{1 - 2i\gamma\mu\omega + \sigma^2\mu\omega^2} - 1)$
Carr-Geman-Madan-Yor	$C\Gamma(-Y)\left[(M-i\omega)^Y - M^Y + (G+i\omega)^Y - G^Y\right]$

PIDE Solution in Fourier Space

- Given the value of $\mathcal{F}[v](t,\omega)$ at time $t_2 \leq T$, the system is easily solved to find the value at time $t_1 < t_2$:

$$\mathcal{F}[v](t_1,\omega) = \mathcal{F}[v](t_2,\omega) \cdot e^{\Psi(\omega)(t_2-t_1)}$$

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- Taking the inverse transform leads to the final result

$$v(t_1,\mathbf{x}) = \mathcal{F}^{-1}\left[\mathcal{F}[v](t_2,\omega) \cdot e^{\Psi(\omega)(t_2-t_1)}\right](\mathbf{x})$$

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- In discrete space, a step backwards is computed via

FST Method

$$\mathbf{v}_{m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}[\mathbf{v}_m] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right]$$

Fourier Space Time-stepping

- European options

$$\mathbf{v}_0 = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\mathbf{v}_1\right] \cdot e^{\Psi(\,\cdot\,)(T-t)}\right]$$

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- Bermudan/American options

$$\mathbf{v}^{\star}_{m-1} = \text{FFT}^{-1}\left[\text{FFT}\left[\mathbf{v}_m\right] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right] ,$$
$$\mathbf{v}_{m-1} = \max\left\{\mathbf{v}^{\star}_{m-1}, \mathbf{v}_M\right\} ,$$

where $\mathbf{v}^{\star}_{m-1}$ represents the holding value of the option

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- Barrier options

$$\mathbf{v}_{m-1} = \text{FFT}^{-1}\left[\text{FFT}\left[\mathbf{v}_m\right] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right] \cdot \mathbb{1}_{\{\mathbf{x}<\mathbf{B}\}} + R \cdot \mathbb{1}_{\{\mathbf{x}\geq\mathbf{B}\}}$$

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- Exotic options such as swing, shout, etc.

American Put Option Results with Penalty Method

N	M	Value	Change	$\log_2$(Ratio)	Time (s)
2048	128	9.22478538			0.027
4096	256	9.22523484	0.0004495		0.109
8192	512	9.22538196	0.0001471	1.6114	0.451
16384	1024	9.22542478	0.0000428	1.7808	1.869
32768	2048	9.22543516	0.0000104	2.0444	8.195

- *Option:* American put option $S = 90.0$, $K = 98.0$, $T = 0.25$
- *Model:* CGMY model $C = 0.42$, $G = 4.37$, $M = 191.2$, $Y = 1.0102$, $r = 0.06$
- *Convergence:* 2 in space and 2 in time
- *Reference price:* 9.2254803 from Forsyth, Wan and Wang 2007

Computation of Greeks

- Delta – $\partial_{S_k} v(t, \mathbf{x})$
  - Differentiation in real space computed via scaling in Fourier space

$$\begin{aligned}\partial_{\mathbf{S}_k} v(t, \mathbf{x}) &= \partial_{\mathbf{x}_k} v(t, \mathbf{x}) / \left(\mathbf{S}_k(0) e^{\mathbf{x}_k}\right) \\ &= \mathcal{F}^{-1}\left[i\omega_k \cdot \mathcal{F}[v](t, \omega)\right](\mathbf{x}) / \left(\mathbf{S}_k(0) e^{\mathbf{x}_k}\right)\end{aligned}$$

  - The discrete method for computing Deltas is then given by

$$\mathbf{\Delta}_{k,m-1} = \text{FFT}^{-1}\left[i\omega_k \cdot \hat{\mathbf{v}}_{m-1}\right] / \left(\mathbf{S}_k(0) e^{\mathbf{x}_k}\right) ,$$

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where $\hat{\mathbf{v}}_{m-1} = \text{FFT}\left[\mathbf{v}_m\right] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}$

- Gamma – $\partial_{S_k^2} v(t, \mathbf{x})$
  - Differentiation in real space computed via scaling in Fourier space

$$\begin{aligned}\partial^2_{\mathbf{S}_k^2} v(t,\mathbf{x}) &= \left(-\partial_{\mathbf{x}_k} + \partial_{\mathbf{x}_k^2}\right) v(t,\mathbf{x}) / \left(\mathbf{S}_k(0) e^{\mathbf{x}_k}\right)^2 \\ &= \mathcal{F}^{-1}\left[(-i\omega_k - \omega_k^2) \cdot \mathcal{F}[v](t,\boldsymbol{\omega})\right](\mathbf{x}) / \left(\mathbf{S}_k(0) e^{\mathbf{x}_k}\right)^2\end{aligned}$$

  - The discrete method for computing Gammas is then given by

$$\boldsymbol{\Gamma}_{k,m-1} = \text{FFT}^{-1}\left[-(i\omega_k + \omega_k^2) \cdot \hat{\mathbf{v}}_{m-1}\right] / \left(\mathbf{S}_k(0) e^{\mathbf{x}_k}\right)^2$$

Computation of Greeks (ctd.)

- Vega – $\partial_{\sigma_k} v(t, \mathbf{x})$
  - Vega satisfies a PIDE

$$\partial_{\sigma_k}\left\{(\partial_t + \mathcal{L})\, v(t, \mathbf{x})\right\} = (\partial_t + \mathcal{L})\, \partial_{\sigma_k} v(t, \mathbf{x}) + \mathcal{H}_{\sigma_k} v(t, \mathbf{x}) = \mathbf{0}\,,$$

    where $\mathcal{H}_{\sigma_k} = (\partial_{\sigma_k}\gamma)'\partial_{\mathbf{x}} + \partial'_{\mathbf{x}}(\partial_{\sigma_k}\mathbf{\Sigma})\partial_{\mathbf{x}}$.
  - The PIDE can be solved explicitly in Fourier space

$$\partial_{\sigma_k} v(t, \mathbf{x}) = (T - t)\, \mathcal{F}^{-1}\left[\mathcal{F}\left[\mathcal{H}_{\sigma_k}\right](\boldsymbol{\omega}) \cdot \mathcal{F}\left[v\right](t, \boldsymbol{\omega})\right](\mathbf{x})\,.$$

  - The greekFST method for computing Vegas is then given by

$$\nabla_{k,m-1} = \Delta t_m \mathrm{FFT}^{-1}\left[\mathcal{F}\left[\mathcal{H}_{\sigma_k}\right](\boldsymbol{\omega}) \cdot \hat{\mathbf{v}}_{m-1}\right]$$

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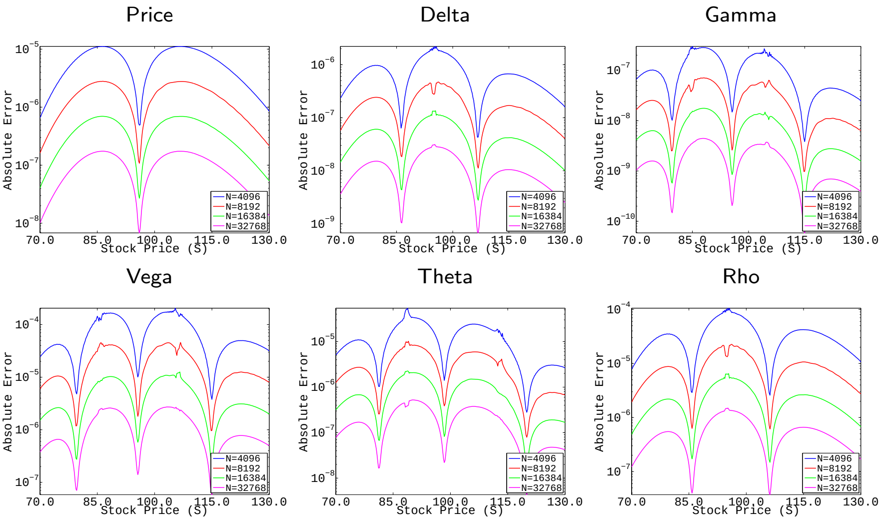
- Rho – $\partial_r v(t,\mathbf{x})$
  - Rho satisfies a similar PIDE and the discrete method for computing Rho is

$$\mathbf{P}_{m-1} = \Delta t_m \text{FFT}^{-1}\left[\mathcal{F}\left[\mathcal{H}_r\right](\boldsymbol{\omega})\cdot\hat{\mathbf{v}}_{m-1}\right]$$

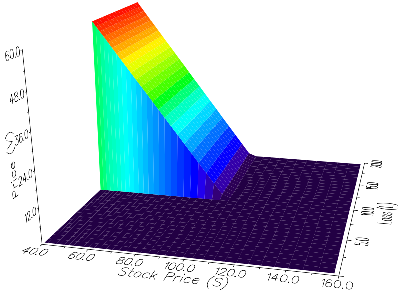
Greeks Errors



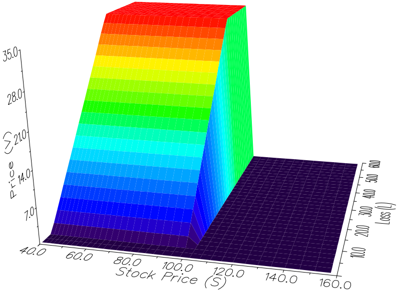
- Digital call option $K = 100$, $T = 0.5$
- Merton jump-diffusion model $\sigma = 0.15$, $r = 0.05$, $q = 0.02$, $\lambda = 0.1$, $\tilde{\mu} = -1.08$, $\tilde{\sigma} = 0.4$

Catastrophe Options

- Catastrophe options pay the holder a function of total losses and the company's equity value



Catastrophe equity put
$\varphi(S,L) = \mathbb{1}_{\{L>L^*\}}(K-S)_+$



Double-trigger stop-loss
$\varphi(S,L) = \mathbb{1}_{\{S<K\}}[(L-L_a)_+ - (L-L_d)_+]$

Catastrophe Options (ctd.)

- Joint model for losses and share price

$$S(t) = S(0)\, e^{\gamma t - \chi\, L(t) + \sigma\, W(t)}\,,$$
$$L(t) = \sum_{n=1}^{N(t)} l_n\,,$$

where χ represents the percentage drop in the share price per unit of loss

- The 2-dimensional Lévy density is $\nu(dy_1 \times dy_2) = f_L(y_2)\,\delta(y_1 + \chi\, y_2)\, dy_1 dy_2$ resulting in the characteristic function

$$\Psi(\omega_1, \omega_2) = i\,\gamma\,\omega_1 - \tfrac{1}{2}\sigma^2\,\omega_1^2 + \lambda \left[\left[1 - \frac{i\, v_l}{m_l}(-\chi\,\omega_1 + \omega_2) \right]^{-\frac{m_l^2}{v_l}} - 1 \right]$$

FST Method Summary

- Various jump-diffusion and exponential Lévy models are handled generically
- Option values are obtained for a range of spot prices
- Can readily price path-dependent and multi-asset options
- Two FFTs per time-step are required
- No time-stepping for European options or between monitoring dates of discretely monitored options
- Second order convergence in space and second order convergence in time for American options with penalty method
- Extendable to computation of the Greeks

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Regime Switching Pricing Framework

- Let $\mathbb{K} \triangleq \{1, \ldots, K\}$ denote the possible states of the world
- Let $Z(t) \in \mathbb{K}$ denote the prevailing state of the world at time t, driven by a continuous time Markov chain with generator $\mathbf{A}$. The transition probability from state k at time t_1 to state l at time t_2 is $P_{kl}^{t_1 t_2} = \exp\{(t_2 - t_1)\mathbf{A}\})_{kl}$
- Given that $Z(t) = k$, the joint stock price process $\mathbf{S}(t)$ is assumed to be following a d-dimensional exponential Lévy model

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- Given that $Z(t) = k$, the joint stock price process $\mathbf{S}(t)$ is assumed to be following a d-dimensional exponential Lévy model
- The discounted-adjusted and log-transformed option prices at time t conditional on the state $Z(t) = k$, denoted by $v^{\langle k \rangle}(t, \mathbf{x})$, satisfy the following system of PIDEs:

$$\begin{cases} \left(\partial_t + A_{kk} + \mathcal{L}^{\langle k \rangle}\right) v^{\langle k \rangle}(t, \mathbf{x}) + \sum_{l \neq k} A_{kl}\, v^{\langle l \rangle}(t, \mathbf{x}) & = \mathbf{0}\,, \\ v^{\langle k \rangle}(t, \mathbf{x}) & = \varphi(\mathbf{S}(0)e^{\mathbf{x}}) \end{cases}$$

Regime Switching Pricing Framework in Fourier Space

- The transformed PIDE can be written in a compact matrix form

$$\begin{cases} (\partial_t + \boldsymbol{\Psi}(\omega))\,\mathcal{F}[\vec{v}](t,\omega) & = 0\,, \\ \mathcal{F}[\vec{v}](T,\omega) & = \mathcal{F}[\varphi](\omega)\,\vec{1}\,, \end{cases}$$

where $\vec{v}$ is the collection of $v^{\langle k \rangle}$'s stacked into a column vector and the elements of the "matrix characteristic function" $\boldsymbol{\Psi}$ are

$$\langle \boldsymbol{\Psi}(\omega) \rangle_{kl} \triangleq \begin{cases} A_{kk} + \Psi^{\langle k \rangle}(\omega), & k = l\,, \\ A_{kl}, & k \neq l \end{cases}$$

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where $\vec{v}$ is the collection of $v^{\langle k \rangle}$'s stacked into a column vector and the elements of the "matrix characteristic function" $\boldsymbol{\Psi}$ are

$$\langle \boldsymbol{\Psi}(\omega) \rangle_{kl} \triangleq \begin{cases} A_{kk} + \Psi^{\langle k \rangle}(\omega), & k = l\,, \\ A_{kl}, & k \neq l \end{cases}$$

- In discrete space, a step backwards is computed via

rsFST Method

$$\vec{v}_{m-1} = \mathrm{FFT}^{-1}\left[e^{\boldsymbol{\Psi}(\Delta t_m)} \cdot \mathrm{FFT}\left[\vec{v}_m\right] \right]$$

Hidden vs. Visible Regimes for Bermudan Options

- The conditional holding price $\vec{\mathbf{v}}^{\star}_{m-1}$ at time step $m-1$ is

$$\vec{\mathbf{v}}^{\star}_{m-1} = \text{FFT}^{-1}\left[e^{\boldsymbol{\Psi}\Delta t} \cdot \text{FFT}\left[\vec{\mathbf{v}}_m\right]\right]$$

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- If the states are visible, the exercise boundary must be obtained independently for each state

$$\left[\vec{\mathbf{v}}_{m-1}\right]^{\langle k \rangle} = \max\left\{\left[\vec{\mathbf{v}}^{\star}_{m-1}\right]^{\langle k \rangle}, \mathbf{v}_M\right\}$$

for each $k \in \mathbb{K}$.

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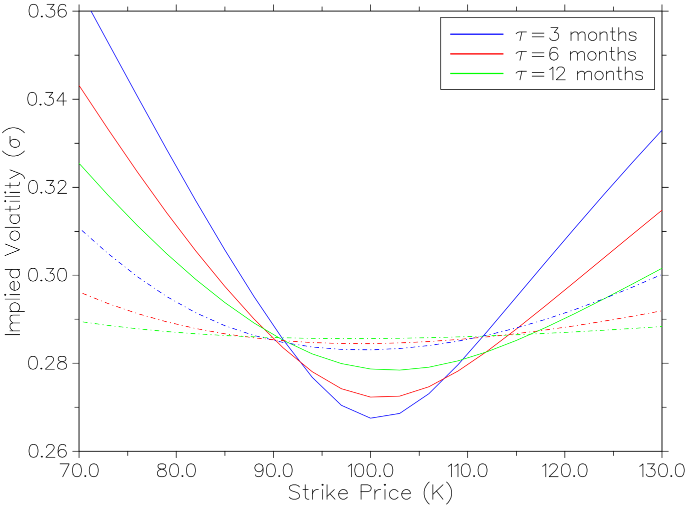
- If the states are hidden, the exercise boundary is the same for every state.

$$\mathbf{v}_{m-1} = \max\left\{\mathbf{v}^{\star}_{m-1}, \mathbf{v}_M\right\}$$

where $\mathbf{v}^{\star}_{m-1}$ is the (single) unconditional holding price

$$\mathbf{v}^{\star}_{m-1} = \exp\{(m-1)\Delta t \mathbf{A}\}\, \vec{p} \cdot \vec{\mathbf{v}}^{\star}_{m-1}$$

Implied Volatility Surface



Merton-jump diffusion model with constant and regime-switching volatility

rsFST Method Summary

- Can incorporate regime-switching on any parameter of the model
- Option values are obtained for a range of spot prices
- No time-stepping for European options or between monitoring dates of discretely monitored options
- Computational complexity scales linearly with the number of states

1. Introduction
2. Fourier Space Time-stepping method
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6. Conclusions

Mean-Reverting Pricing Framework

The Model

$$\begin{aligned} d\mathbf{Y}(t) &= -\boldsymbol{\kappa}\mathbf{Y}(t-)dt + d\mathbf{J}(t)\,, \\ \mathbf{X}(t) &= \boldsymbol{\theta} + \boldsymbol{\Lambda}\mathbf{Y}(t)\,, \\ \mathbf{S}(t) &= \mathbf{S}(0)e^{\mathbf{X}(t)}\,, \end{aligned}$$

where $\mathbf{J}(t)$ is a Lévy process with characteristic triplet $(\gamma, \boldsymbol{\Sigma}, \boldsymbol{\nu})$

Mean-Reverting Pricing Framework

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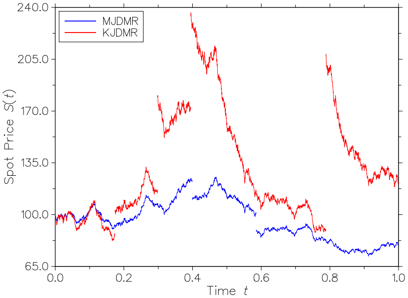
$$\mathbf{S}(t) = \mathbf{S}(0)e^{\mathbf{X}(t)}\,,$$

where $\mathbf{J}(t)$ is a Lévy process with characteristic triplet $(\gamma, \boldsymbol{\Sigma}, \boldsymbol{\nu})$

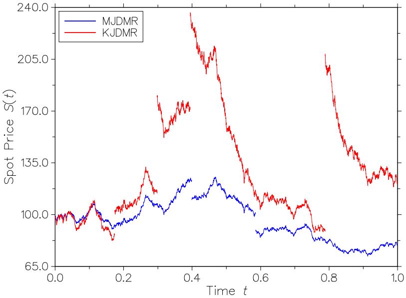
Generalizes a number of well-known models

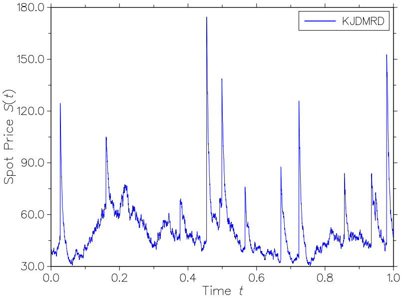
- Gibson and Schwartz (1990)
- Clewlow and Strickland (2000)
- Barlow, Gusev and Lai (2004)
- Hikspoors and Jaimungal (2007)

Examples of obtainable spot-price dynamics

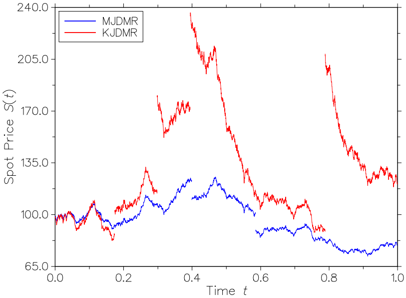


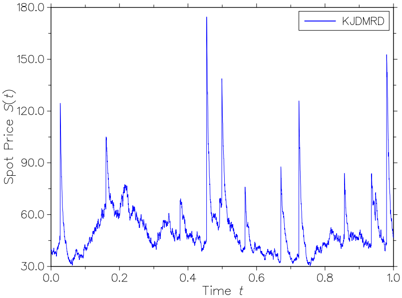
Examples of obtainable spot-price dynamics

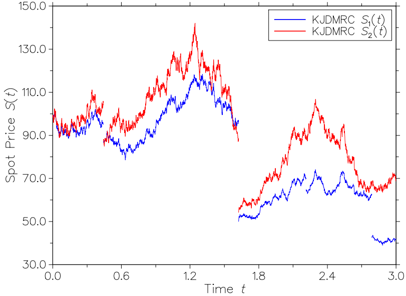




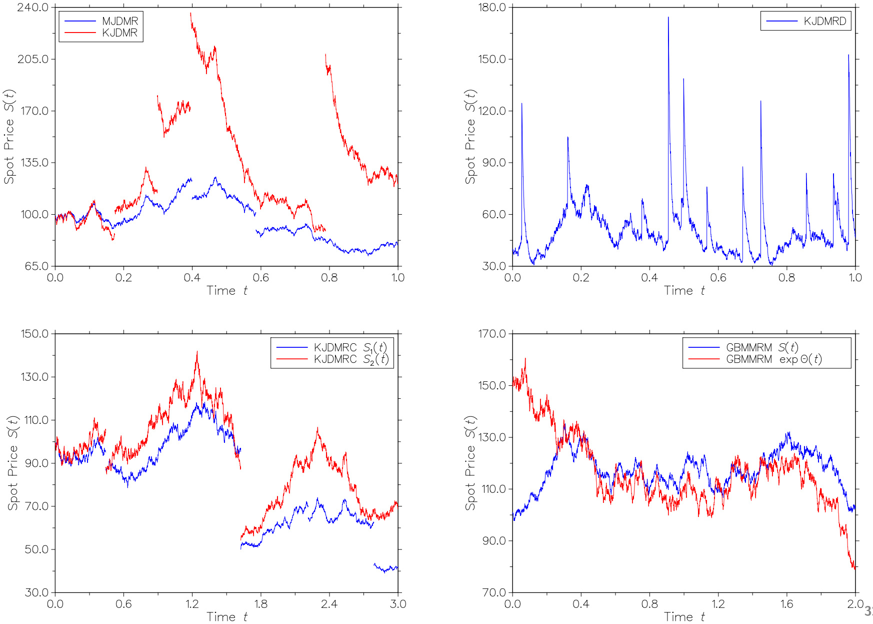
Examples of obtainable spot-price dynamics







Examples of obtainable spot-price dynamics



PIDE Transformation into Fourier Space

- The option price satisfies a PIDE

$$\begin{cases} (\partial_t + \mathcal{L})\, v(t, \mathbf{y}) & = 0\,, \\ v(T, \mathbf{y}) & = \varphi(\mathbf{S}(0) e^{\boldsymbol{\theta} + \boldsymbol{\Lambda}\, \mathbf{y}})\,, \end{cases}$$

where $\mathcal{L}$ is the infinitesimal generator of the $\mathbf{Y}(t)$ process:

$$\mathcal{L} f(\mathbf{y}) = ((\boldsymbol{\kappa}(\boldsymbol{\theta} - \mathbf{y}))' \partial_{\mathbf{y}} + \tfrac{1}{2} \partial_{\mathbf{y}}' \boldsymbol{\Sigma} \partial_{\mathbf{y}}) f(\mathbf{y}) + \int [f(\mathbf{y} + \mathbf{z}) - f(\mathbf{y})]\, \nu(d\mathbf{z})$$

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- The PIDE is transformed into a d-parameter family of ODEs parameterized by $\boldsymbol{\omega}$:

$$\begin{cases} (\partial_t + \psi(e^{-\boldsymbol{\kappa}'(T-t)}\boldsymbol{\omega}) + \mathrm{Tr}\,\boldsymbol{\kappa})\, \mathcal{F}[\tilde{v}](t, \boldsymbol{\omega}) & = 0\,, \\ \mathcal{F}[\tilde{v}](T, \boldsymbol{\omega}) & = \mathcal{F}[\tilde{\varphi}](\boldsymbol{\omega})\,, \end{cases}$$

where

$$\mathcal{F}[\tilde{v}](t, \boldsymbol{\omega}) = \mathcal{F}[v](t, e^{-\boldsymbol{\kappa}'(T-t)}\, \boldsymbol{\omega})$$

Fourier Space Solution

- The ODE can be solved in Fourier space

$$\mathcal{F}[v](t_1, \boldsymbol{\omega}) = \mathcal{F}[v](t_2, e^{\boldsymbol{\kappa}'(t_2-t_1)}\, \boldsymbol{\omega}) \cdot e^{\Psi(t_2-t_1,\boldsymbol{\omega})+(t_2-t_1)\mathrm{Tr}\,\boldsymbol{\kappa}}$$

where

$$\Psi(t_2 - t_1, \boldsymbol{\omega}) = \int_0^{t_2-t_1} \psi(e^{\boldsymbol{\kappa}' u}\, \boldsymbol{\omega})\, du\ , \qquad \text{and}$$

$$\psi(\boldsymbol{\omega}) = -\tfrac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} + \int \left(e^{i\boldsymbol{\omega}'\mathbf{y}} - 1\right) \nu(d\mathbf{y})$$

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- The real-space solution is

$$v(t_1, \mathbf{y}) = \mathcal{F}^{-1}\left[\mathcal{F}[v](t_2, e^{\boldsymbol{\kappa}'(t_2-t_1)}\,\boldsymbol{\omega}) \cdot e^{\Psi(t_2-t_1,\boldsymbol{\omega})+(t_2-t_1)\mathrm{Tr}\,\boldsymbol{\kappa}}\right](\mathbf{y})$$

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- In discrete space, a step backwards is computed via

mrFST Method

$$\mathbf{v}_{m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\check{\mathbf{v}}_m\right] \cdot e^{\Psi(\Delta t_m,\,\cdot\,)}\right]\;, \quad \text{where } \check{v}(\mathbf{y}) \triangleq v(\mathbf{y}\,e^{-\kappa'\Delta t})$$

Discrete Barrier Option Results

N	M	Value	Change	$\log_2$Ratio	Time (sec.)
2048	126	0.58232029			0.033
4096	126	0.58303190	0.0007116		0.067
8192	126	0.58244527	0.0005866	0.2786	0.156
16384	126	0.58232817	0.0001171	2.3248	0.318
32768	126	0.58231749	0.0000107	3.4541	0.674

- *Option:* Up-and-out barrier call $S = 100, K = 105, T = 0.5,$ $B = 115, R = 0.5$ with daily monitoring
- *Model:* Kou jump-diffusion with mean reversion $\sigma = 0.3, \lambda = 4.0, \eta_p = 0.95, \eta_+ = 0.3, \eta_- = 0.1, \theta = 92.0, \kappa = 5.0, r = 0.06$
- *Monte Carlo:* 0.58289924 – 95% CI width of 0.0028937 @ 56 sec.

mrFST Method Summary

- Readily extends from the FST method using interpolation of option prices grid
- Retains all of the computational efficiency of the FST method
- For high mean-reversion rates, several time-steps may be required

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Pricing Framework under Risk-adjusted Measure $\mathbb{Q}$

The Model

$$r(t) = X_1(t) + \ldots + X_d(t) + \delta(t)$$

where $\mathbf{X}(t) = [X_1(t), \ldots, X_d(t)]'$ satisfy

$$d\mathbf{X}(t) = -\boldsymbol{\kappa}\mathbf{X}(t)dt + d\mathbf{J}(t), \quad \mathbf{X}(0) = \mathbf{0}$$

and $\delta(t)$ is a deterministic function chosen to fit the currently-observed term structure of bond prices.

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The option price satisfies a PIDE

$$\begin{cases} (\partial_t + \mathcal{L})\, V_Q(t, \mathbf{x}) & = 0\,, \\ V_Q(T, \mathbf{x}) & = \varphi(x)\,, \end{cases}$$

where $\mathcal{L}$ is

$$\mathcal{L}f(\mathbf{x}) = (-\boldsymbol{\kappa}\mathbf{x}'\partial_\mathbf{x} + \tfrac{1}{2}\partial_\mathbf{x}'\boldsymbol{\Sigma}\partial_\mathbf{x})f(\mathbf{x}) + \int [f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x})]\,\nu(d\mathbf{z}) - (\Sigma \mathbf{x}_i + \delta(t))$$

Pricing Framework under T-forward Measure $\mathbb{T}$

The Model

$$r(t) = X_1(t) + \ldots + X_d(t) + \delta(t)$$

where $\mathbf{X}(t) = [X_1(t), \ldots, X_d(t)]'$ satisfy

$$d\mathbf{X}(t) = -(\boldsymbol{\kappa}\mathbf{X}(t) + \boldsymbol{\Sigma}\mathbf{B}(t))dt + d\mathbf{J}(t), \quad \mathbf{X}(0) = \mathbf{0}$$

and $\mathbf{B}(t) = [B_1(t), \ldots, B_d(t)]'$, $B_i(t) = (1 - e^{-\kappa_i(T-t)})/\kappa_i$

Pricing Framework under T-forward Measure $\mathbb{T}$

The Model

$$r(t) = X_1(t) + \ldots + X_d(t) + \delta(t)$$

where $\mathbf{X}(t) = [X_1(t), \ldots, X_d(t)]'$ satisfy

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The option price satisfies a PIDE

$$\begin{cases} (\partial_t + \mathcal{L})\, V_T(t, \mathbf{x}) & = 0\,, \\ V_T(T, \mathbf{x}) & = \varphi(x)\,, \end{cases}$$

where $\mathcal{L}$ is

$$\mathcal{L}f(\mathbf{x}) = (-\boldsymbol{\kappa}\mathbf{x}'\partial_\mathbf{x} - \partial_\mathbf{x}'\boldsymbol{\Sigma}\mathbf{B}(t) + \tfrac{1}{2}\partial_\mathbf{x}'\boldsymbol{\Sigma}\partial_\mathbf{x})f(\mathbf{x}) + \int [f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x})]\, e^{\mathbf{B}(t)\mathbf{z}}\nu(d\mathbf{z})$$

Pricing Framework under T-forward Measure $\mathbb{T}$

The Model

$$r(t) = X_1(t) + \ldots + X_d(t) + \delta(t)$$

where $\mathbf{X}(t) = [X_1(t), \ldots, X_d(t)]'$ satisfy

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Now we can apply the results from mrFST method!

Fourier Space Solution

- The solution to the PIDE in Fourier space is

$$V_T(t_1, \mathbf{x}) = \mathcal{F}^{-1}\left[\mathcal{F}[V_T](t_2, e^{\boldsymbol{\kappa}'(t_2-t_1)}\,\boldsymbol{\omega}) \cdot e^{\Psi(t_2-t_1,\boldsymbol{\omega})+(t_2-t_1)\mathrm{Tr}\,\boldsymbol{\kappa}}\right](\mathbf{x})$$

where

$$\Psi(t_1, t_2, \boldsymbol{\omega}) = \int_{t_1}^{t_2} \psi(u, e^{\boldsymbol{\kappa}'(u-t_1)}\,\boldsymbol{\omega})\,du\,, \qquad \text{and}$$

$$\psi(t, \boldsymbol{\omega}) = -i\boldsymbol{\omega}\boldsymbol{\Sigma}\mathbf{B}(t) - \tfrac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} + \int \left(e^{i\boldsymbol{\omega}'\mathbf{z}} - 1\right) e^{\mathbf{B}(t)\mathbf{z}}\nu(d\mathbf{z})\,,$$

and

$$V_Q(t, \mathbf{x}) = V_T(t, \mathbf{x}) \cdot P(t, T, \mathbf{x})$$

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and

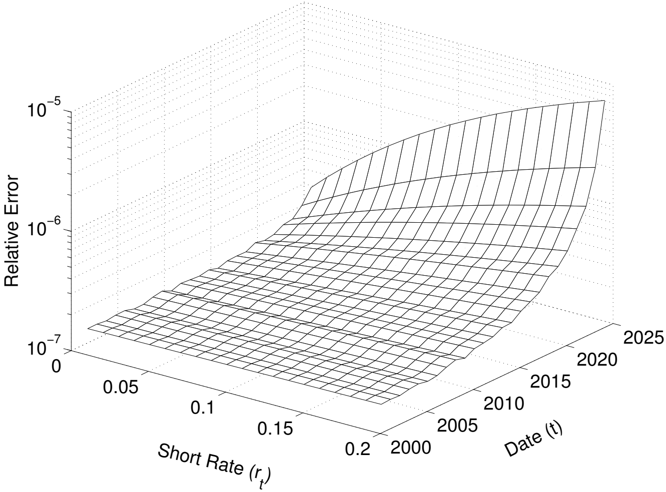
$$V_Q(t, \mathbf{x}) = V_T(t, \mathbf{x}) \cdot P(t, T, \mathbf{x})$$

- In discrete space, a step backwards is computed via

irFST Method

$$\mathbf{V}_{T,m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\check{\mathbf{V}}_{T,m}\right] \cdot e^{\Psi(t_{m-1},t_m,\,\cdot\,)}\right]\,, \text{ where } \check{v}_T(\mathbf{y}) \triangleq v_T(\mathbf{y}\, e^{-\kappa'\Delta t})$$

Numerical Results



Relative error for pricing a 5-year swap under the Hull-White model

irFST Method Summary

- Currently, handles the following models
  1-factor: Vasicek, Hull-White, and Vasicek-J++
  2-factor: Hull-White 2F, G2++, and G2-J++
- Readily extends from the FST method using interpolation of option prices grid
- Retains all of the computational efficiency of the FST method
- For high mean-reversion rates, several time-steps may be required

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FST Framework Summary

The Approach

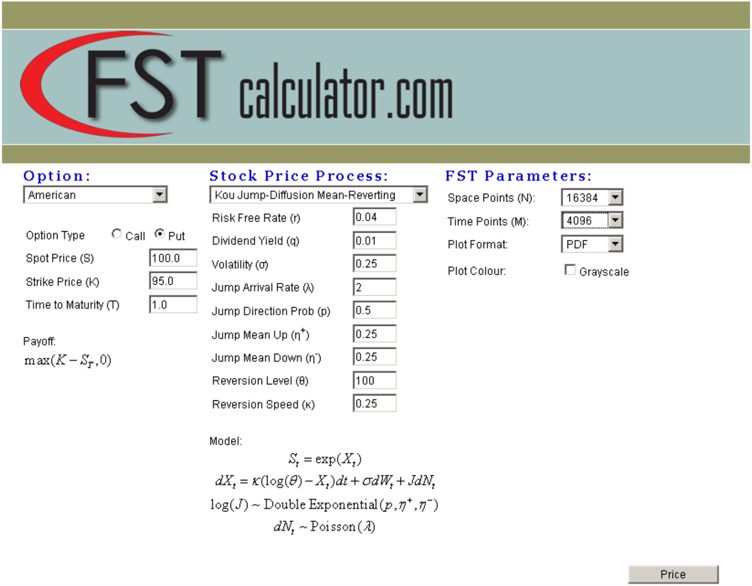
- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space and solve the ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

FST Framework Summary

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space and solve the ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

- Various jump-diffusion and exponential Lévy models are handled generically
- Option values are obtained for a range of spot prices
- Can readily price path-dependent and multi-asset options
- Two FFTs per time-step are required
- No time-stepping for European options or between monitoring dates of discretely monitored options
- Second order convergence in space and second order convergence in time for American options with penalty method



MATLAB code is also available

Thank You!

Jackson, K. R., S. Jaimungal, and V. Surkov (2007).
Option pricing with regime switching Lévy processes using Fourier space time-stepping.
In *Proceeding of the Fourth IASTED International Conference on Financial Engineering and Applications*, pp. 92–97.

Jackson, K. R., S. Jaimungal, and V. Surkov (2008).
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Journal of Computational Finance 12(2), 1–28.

Jaimungal, S. and V. Surkov (2008).
A general Lévy-based framework for energy price modeling and derivative valuation via FFT.
Working paper, available at `http://ssrn.com/abstract=1302887`.

More at `http://ssrn.com/author=879101`