

Markovian projection on displaced diffusion.

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Outline

- Theory
- Review of the BGM model
- LMM swaption formula
- CC LMM/BS/LMM approximation of FX-options

Mathematical challenges

- Complexity of modern models
- Sensitivity of the instruments to distant wings of volatility surfaces (wide range of European option strikes)



Requirements to European option pricing:

- Effective methods for complicated underlying process
- Good approximation quality for all strikes, capability to reproduce model skew/smile

European option on a generic rate process

Rate process (in general non-Markovian)

$$dS(t) = \Sigma(t) \cdot dW(t), \quad S(0) = S_0$$

with a generic volatility (vector) process $\Sigma(t)$ driven by F independent Brownian motions, $W(t) = \{W_1(t), \dots, W_F(t)\}$. For example, S can be a swap rate, discounted FX-rate, etc.

The process $S(t)$ underlies the European option $E[(S(t) - K)^+]$ maturing at t

→ for calculations we need only its marginal distribution at t

Markovian projection

Term "Markovian projection" was coined by Piterbarg; idea ascends to Gyöngy (1986) and Dupire (1994).

A complicated, non-Markovian process is replaced by a Markovian one, s.t. both processes have identical one-dimensional marginal distributions. By Gyöngy's lemma, the *Markovian* process $S^*(t)$ satisfying

$$dS^*(t) = \Sigma^*(t, S^*(t)) \cdot dW, \quad S^*(0) = S_0,$$

with

$$|\Sigma^*(t, s)|^2 = E[|\Sigma(t)|^2 \mid S(t) = s]$$

has the same marginal distributions as $S(t)$ for any t , and therefore can be used to compute the European option,

$$E[(S(t) - K)^+] = E[(S^*(t) - K)^+]$$

To avoid complicated *conditional* expectation $E[|\Sigma(t)|^2 | S(t) = s]$ calculus one postulates a *low parametric* mimicking process $S^*(t)$

$$dS^*(t) = \Sigma^*(t, S^*(t), p_1(t), p_2(t), \dots) \cdot dW,$$

and calculate its optimal parameters via *minimization of non-conditional expectations*.

The mimicking process choice depends on characteristics of initial process $S(t)$ (skew, smile).

Examples

- Piterbarg (2005b), Antonov & Misirpashaev (2006a):
Projection on a displaced diffusion (DD) with time-dependent parameters for the expectation
- Piterbarg (2006ab), Antonov, Misirpashaev and Piterbarg (2007):
Projection on a Heston model capturing skew and smile.
Application: calibration of basket of Heston models
- Antonov, Arneguy and Audet (2008)
Projection on affine generalization of the Heston model capturing skew and smile.
Application: calibration of correlated cross-currency
HW/HW/Heston

Markovian projection on a displaced diffusion (DD)

Capturing the first derivative of effective local vol

$$\Sigma^*(t, s) \simeq (1 + \Delta s \beta(t)) \sigma(t), \quad \Delta s = s - S_0$$

The first derivative \Leftrightarrow implied volatility skew

Good approximation accuracy for non-SV models exhibiting a skew

Markovian projection on DD: method

Displaced diffusion

$$dS^*(t) = (1 + \Delta S^*(t)\beta(t)) \sigma(t) \cdot dW(t), \quad S^*(0) = S_0$$

as a mimicking process for the initial one $dS(t) = \Sigma(t) \cdot dW(t)$.

Here $\Delta S^*(t) = S^*(t) - S_0$, $\sigma(t)$ is an F -component deterministic volatility vector, and $\beta(t)$ is a time-dependent shift (controlling skew).

We look for optimal $\sigma(t)$ and $\beta(t)$, s.t. for any t ,

$$|\sigma(t)|^2 (1 + \Delta s \beta(t))^2 \simeq E[|\Sigma(t)|^2 | S(t) = s]$$

holds to the highest order in volatilities.

Projections vs. conditional expectation

Denote the true conditional expectation,

$$|\Xi(t, x)|^2 = E[|\Sigma(t)|^2 \mid S(t) = x]$$

For every fixed t , the conditional expectation can be characterized as a function of state $|\Xi(t, x)|^2$ that minimizes the L_2 -distance from the true variance,

$$\chi^2 = E \left[(|\Sigma(t)|^2 - |\Xi(t, S(t))|^2)^2 \right] \rightarrow \min.$$

DD case: minimization of

$$\chi^2 = E \left[(|\Sigma(t)|^2 - |\sigma(t)|^2 (1 + \Delta S(t)\beta(t))^2)^2 \right]$$

for each t gives a condition for optimal DD coefficients $\sigma(t)$ and $\beta(t)$.

DD results

Optimal DD coefficients in terms of unconditional averages,

$$|\sigma(t)|^2 = E [|\Sigma(t)|^2] + O(\varepsilon^4)$$

$$\beta(t) = \frac{E [|\Sigma(t)|^2 \Delta S(t)]}{2 E [|\Sigma(t)|^2] E [\Delta S^2(t)]} + O(\varepsilon^2)$$

where ε is a scale of volatilities

Option price with maturity T can be found from the BS formula after the shift $\beta(t)$ is averaged (Piterbarg)

$$\bar{\beta}_T = \frac{\int_0^T \beta(t) |\sigma(t)|^2 \int_0^t |\sigma(\tau)|^2 d\tau dt}{\int_0^T |\sigma(t)|^2 \int_0^t |\sigma(\tau)|^2 d\tau dt}$$

Closed-form results can be obtained using small volatilities expansions of the above expectations.

It is technically convenient to introduce so called *separable* processes with following features:

- expectations underlying above optimal formulas are available for the separable process in a leading order in small vols
- most of processes underlying european options can be presented as a separable process

Separable processes

The rate process satisfies

$$dS(t) = \sum_n X_n(t) a_n(t) \cdot dW(t),$$

where $a_n(t)$ are deterministic vector functions with scalar components $a_{n,\nu}(t)$ ($\nu = 1, \dots, F$), and $X_n(t)$ are stochastic processes obeying SDEs of the form

$$dX_n(t) = \mu_n(t, \{X_k(t)\})dt + \sigma_n(t, \{X_k(t)\}) \cdot dW(t).$$

We assume the drift terms μ_n are small in the sense that they are of the second or higher order in volatilities,

$$\mu_n = O(\sigma_+^2)$$

where

$$\sigma_+ = \max\{\sigma_{n,\nu}(t, \{X_k(t)\}), a_{n,\nu}(t)\}$$

Optimal DD coefficients for a separable process

$$\sigma(t) = \sum_n X_n(0) a_n(t) + O(\sigma_+^3),$$

$$\beta(t) = \frac{\sum_n (a_n(t) \cdot \sigma(t)) \int_0^t (\sigma_n(\tau, \{X_k(0)\}) \cdot \sigma(\tau)) d\tau}{|\sigma(t)|^2 \int_0^t |\sigma(\tau)|^2 d\tau} + O(\sigma_+^2).$$

Many challenging calibration problems (not involving SV) can be easily resolved by separable models approximations.

Technical details and multiple applications can be found in Antonov & Misirpashaev (2006a).

Applications of the MP to DD

- Shifted BGM swaption formula
Piterbarg (2006b); Antonov & Misirpashaev (2006a)
- Cross-Currency models (CEV model for FX-rate, gaussian model for interest rates)
Piterbarg (2005b); Antonov & Misirpashaev (2006a)
- LMM Cross-Currency models (CEV for FX-rate, shifted BGM for interest rates)
Antonov & Misirpashaev (2006a), (2006b)

Example: projecting a basket of DD models

Initial process is a weighted sum

$$S(t) = \sum_i w_i S_i(t)$$

of DD models

$$dS_i(t) = (1 + \Delta S_i(t)\beta_i) \lambda_i \cdot dW(t)$$

For simplicity we take time-independent parameters, although the approximation works for time-dependent one.

Representing the DD basket as a separable process, we obtain effective volatility and skew parameter.

Basket of DD models as a separable process

$$dS(t) = \sum_i w_i dS_i(t)$$

$$\Downarrow$$

$$dS(t) = \sum_n X_n(t) a_n(t) = \left(\sum_i w_i \lambda_i + \Delta S_i(t) w_i \beta_i \lambda_i \right) \cdot dW(t)$$

This immediately gives the optimal DD parameters

$$\sigma(t) = \sigma \simeq \sum_i w_i \lambda_i$$

$$\beta(t) = \beta \simeq \frac{\sum_i w_i \beta_i (\lambda_i \cdot \sigma)^2}{|\sigma|^4}$$

Error cancelation

For the basket containing a *single* asset, the MP procedure restores *exactly* the initial DD process in spite of two approximations:

- leading order approximation for the optimal DD coefficients
- leading order approximation for the underlying averages

Remark. For a given time-horizon one can approximate the DD basket by a single DD process using exact moment fit ($E[S^2(T)]$ and $E[S^3(T)]$). Pricing with this process has often similar accuracy as the straightforward MP.

Numerical experiments: MP to DD

Basket of 5 assets with the following parameters of DD

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
vol of rate (%)	14	15	16	17	18
beta skew (%)	30	40	50	60	70

- Initial values $S_i(0) = 1$ and weights $w_i = 1/5$ for all i
- Correlations between rates: $\frac{\lambda_i \cdot \lambda_j}{|\lambda_i| |\lambda_j|} = 70\%$ for all i, j

The optimal values are

$$|\sigma| = 13.95\%$$

$$\beta = 63.24\%$$

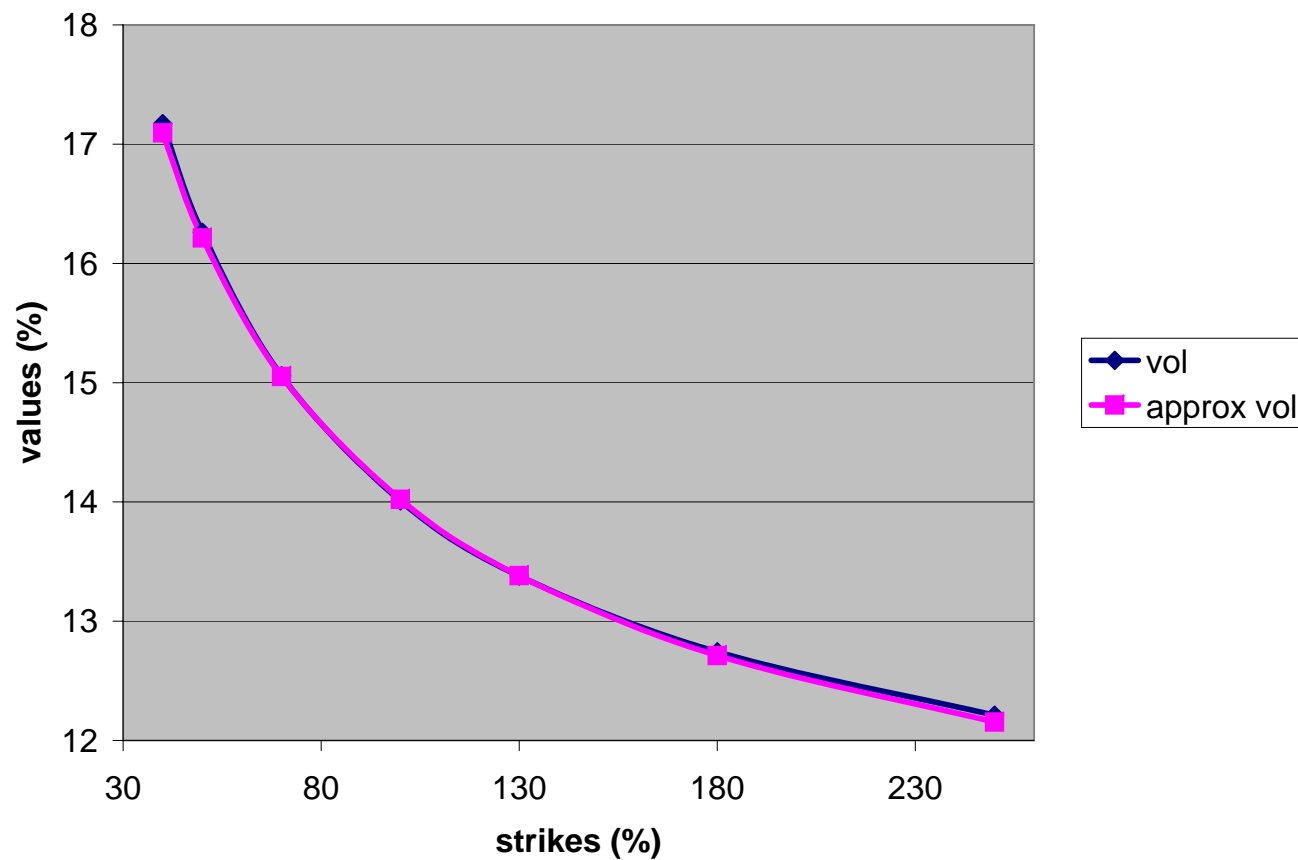


Figure 1: DD basket option implied volatilities for 10Y maturity

Review of the Libor market model (LMM).

The F -factor LMM (BGM, Shifted BGM)

- set of maturities T_n , $n = 0, \dots, N$ such that $0 < T_0 < T_1 < \dots < T_N$
- initial forward LIBOR rates $L_n(0)$
- F independent Brownian motions $W_f(t)$ for $f = 1, \dots, F$
- volatility vectors $\gamma_n(t) = \{\gamma_{n,1}(t) \cdots \gamma_{n,F}(t)\}$, $n = 0, \dots, N - 1$
- shifts b_n , $n = 0, \dots, N - 1$

Forward Libor $L_n(t)$ is a forward rate starting at T_n and ending at T_{n+1}

$$L_n(t) = \frac{1}{\delta_n} \left(\frac{P(t, T_n)}{P(t, T_{n+1})} - 1 \right)$$

Libors SDE's.

$$\begin{aligned} dL_n(t) &= \dots + (L_n(t) b_n + L_n(0) (1 - b_n)) \sum_{f=1}^F \gamma_{n,f}(t) dW_f(t) \\ &= \dots + (L_n(t) b_n + L_n(0) (1 - b_n)) \gamma_n(t) \cdot dW(t) \end{aligned}$$

Parameters rescale.

For the BGM model it is more convenient to rescale the DD mimicking process

$$dS^*(t) = (1 + \Delta S^*(t)\beta(t)) \sigma(t) \cdot dW(t)$$

⇓

$$dS^*(t) = (S^*(t)\beta(t) + (1 - \beta(t))S_0) \sigma(t) \cdot dW(t)$$

obtained by

$$\beta(t) \rightarrow \frac{\beta(t)}{S_0}, \quad \sigma(t) \rightarrow \sigma(t) S_0$$

The separable process MP formulas remain *the same* if we rescale $a_n(t) \rightarrow a_n(t) S_0$ to obtain

$$dS(t) = S_0 \sum_n X_n(t) a_n(t) \cdot dW(t)$$

European swaptions.

Forward swap rate $S(t)$ with first fixing at T_B and last payment at T_E

$$S(t) = \frac{P(t, T_B) - P(t, T_E)}{\sum_{i=B+1}^E \delta_{i-1} P(t, T_i)}.$$

A payer swaption price with a fixed rate K and exercise $T < T_B$ expressed via an expectation in the martingale measure for the process $S(t)$,

$$\text{Swaption}(T, K) = E[(S(T) - K)^+] \sum_{i=B+1}^E \delta_{i-1} P_i,$$

where we introduced a notation for bond values at the origin $P_j = P(0, T_j)$.

Swap rate SDE.

The forward swap rate $S(t)$ is a deterministic function of the forward Libors

$$S(t) = S(\{L_i(t)\}) = \frac{\prod_{i=B}^{E-1} (1 + \delta_i L_i(t)) - 1}{\sum_{i=B+1}^E \delta_{i-1} \prod_{j=i}^{E-1} (1 + \delta_j L_j(t))}.$$

Its SDE in the martingale measure

$$dS(t) = \sum_n \left(b_n \gamma_n(t) (L_n - l_n) \frac{\partial S}{\partial L_n} + \gamma_n(t) l_n \frac{\partial S}{\partial L_n} \right) \cdot dW(t)$$

where $l_n \equiv L_n(0)$

In the r.h.s., we recognize the desired sum of the form

$S_0 \sum_n a_n(t) X_n(t)$ with two types of terms (odd and even indexes)

$$a_{2n+1}(t) = b_n \gamma_n(t), \quad X_{2n+1}(t) = \frac{L_n - l_n}{S_0} \frac{\partial S}{\partial L_n},$$

$$a_{2n}(t) = \gamma_n(t), \quad X_{2n}(t) = \frac{l_n}{S_0} \frac{\partial S}{\partial L_n}.$$

Diffusion terms frozen at the starting values of the processes $X_n(t)$ are easily found,

$$dX_{2n+1}(t) = \frac{1}{S_0} \frac{\partial S_0}{\partial l_n} l_n \gamma_n(t) \cdot dW(t) + O(\sigma_+^2),$$

$$dX_{2n}(t) = \frac{l_n}{S_0} \sum_m \frac{\partial^2 S_0}{\partial l_n \partial l_m} l_m \gamma_m(t) \cdot dW(t) + O(\sigma_+^2).$$

Optimal coefficients

$$\sigma(t) = \sum_n \frac{l_n}{S_0} \frac{\partial S_0}{\partial l_n} \gamma_n(t) = \sum_n \frac{\partial(\ln S_0)}{\partial(\ln l_n)} \gamma_n(t),$$

$$\beta(t) = \frac{\sum_n \left(\frac{1}{2} \frac{\partial |\sigma(t)|^2}{\partial(\ln l_n)} + \frac{\partial(\ln S_0)}{\partial(\ln l_n)} (|\sigma(t)|^2 - (1 - b_n) D_n(t)) \right) \int_0^t D_n(\tau) d\tau}{|\sigma(t)|^2 \int_0^t |\sigma(\tau)|^2 d\tau}$$

where $D_n(t) = \sigma(t) \cdot \gamma_n(t)$.

Swaption approximation in the BGM model

$$dS(t) \simeq (S(t)\beta(t) + (1 - \beta(t) S_0)) \sigma(t) \cdot dW(t)$$

Stochastic volatility generalization.

BGM SV Libor SDE

$$dL_n(t) = \dots + (L_n(t) b_n + L_n(0) (1 - b_n)) \sqrt{z(t)} \gamma_n(t) \cdot dW(t)$$

where the SV process $z(t)$ follows the CIR evolution

$$dz(t) = \theta(t)(1 - z(t)) + \sqrt{z(t)} \Gamma(t) dU(t), \quad z(0) = 1$$

with mean-reversion $\theta(t)$, vol-of-vol $\Gamma(t)$ and Brownian motion $U(t)$ **independent** of the Brownian motion $W(t)$ driving the Libors.

The SV $z(t)$ is a common factor for all Libors. It is logical to project the swap rate

$$dS(t) = \sqrt{z(t)} \sum_n \left(\beta_n(t) \gamma_n(t) (L_n - l_n) \frac{\partial S}{\partial L_n} + \gamma_n(t) l_n \frac{\partial S}{\partial L_n} \right) \cdot dW(t)$$

to DD with the *same* SV (shifted Heston model) \rightarrow
one can *reuse* the optimal coefficients obtained for the non-SV BGM.

Swaption approximation in the SV BGM model

$$dS(t) \simeq (S(t)\beta(t) + (1 - \beta(t) S_0)) \sqrt{z(t)} \sigma(t) \cdot dW(t)$$

Numerical experiments

LMM Setup (3 factors)

→ 30Y annual model tenor $T_i = iY$

→ continuous interest rate linearly growing from 4.5% at 1Y to 5% at 30Y (log-linear interpolation for zero bonds)

→ flat vector volatility module $|\gamma_i(t)| = |\gamma_i| = 0.17$

→ flat skew $b_i = \frac{1}{2}$

→ SV (CIR) flat parameters: vol-of-vol $\Gamma(t) = \Gamma = 1$ and mean-reversion $\theta(t) = \theta = 0.2$

Libor correlations

→ Start with the full rank-correlation matrix

$$C_{ij} = \frac{\gamma_i \cdot \gamma_j}{|\gamma_i| |\gamma_j|} = e^{-0.1 |T_i - T_j|}$$

→ Compress it to rank 3 (choose three largest eigenvalues and zero the others; normalize the diagonal)

We compare BS implied vols of

- 10Y10 European call with annual tenor
- 20Y10 European call with annual tenor

calculated by the MP ("Analytics") with the full MC simulations ("MC") for a large variety of strikes

10Y10 and 20Y10 results

exercise	strike (%)	implied vol values (%)		error (bp)
		Analytics	MC	
10	2.72	19.94	19.87	7.0
10	3.18	18.45	18.39	6.4
10	3.73	17.12	17.06	5.4
10	4.37	16.00	15.94	5.2
10	5.11	15.19	15.13	6.4
10	5.99	14.78	14.70	7.7
10	7.01	14.73	14.64	8.8
10	8.22	14.93	14.84	9.4
10	9.62	15.28	15.16	12.6
20	2.24	21.72	21.63	9.4
20	2.80	19.80	19.71	8.9
20	3.50	18.11	18.01	10.1
20	4.38	16.68	16.57	11.4
20	5.47	15.58	15.46	12.2
20	6.85	14.85	14.71	14.6
20	8.56	14.50	14.32	17.4
20	10.71	14.44	14.22	21.5
20	13.39	14.58	14.31	26.8

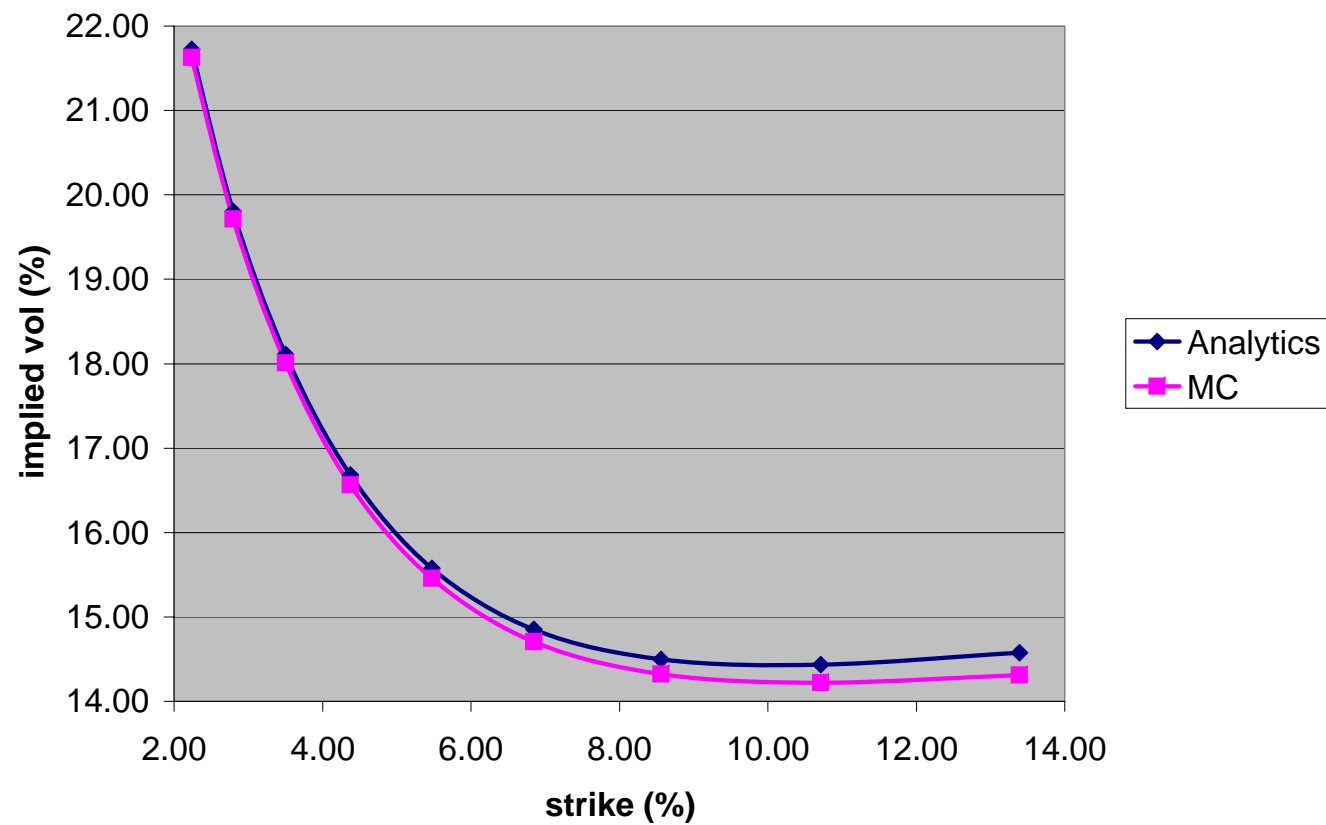


Figure 2: 20Y10 swaption implied vol

Observations/remarks

- Good approximation quality for the optimal analytics: the biggest error does not exceed 13 bps (ATM) and 26 bps (far out-of-money).
- For non-SV cases, the classical approximation (Brace, Gatarek, Musiela (1997)) works often slightly better than the MP, but its generalization to the non-trivial SV rests challenging

Cross-Currency model.

IR models \rightarrow shifted BGM

FX model \rightarrow Log-normal

(see Schlögl (1999) for details)

Our goal is to approximate FX-options price.

Bond ratios.

DD for the forward Libors

$$dL_n(t) = \dots + (L_n(t) b_n + L_n(0) (1 - b_n)) \gamma_n(t) \cdot dW(t)$$

Denote bond ratios (a measure change process between two corresponding forward measures)

$$R_n(t) = \frac{P(t, T_n)}{P(t, T_{n+1})}$$

or

$$R_n(t) = 1 + \delta_n L_n(t)$$

Bond ratios will also follow a displaced diffusion.

Bond ratios SDE's.

$$dR_n(t) = \dots + (\beta_n R_n(t) + (1 - \beta_n) R_n(0)) \sigma_n(t) \cdot dW$$

for volatility

$$\sigma_n(t) = (1 - R_n^{-1}(0)) \gamma_n(t)$$

and shift

$$\beta_n = \frac{b_n}{1 - R_n^{-1}(0)}$$

Denote a formal bond ratios log vol

$$\alpha_n(t) = \left(\beta_n + (1 - \beta_n) \frac{R_n(0)}{R_n(t)} \right) \sigma_n(t)$$

giving

$$dR_n(t) = \dots + R_n(t) \alpha_n(t) \cdot dW$$

The rolling spot (RS) numeraire.

Defined on a discrete tenor $\{T_0, T_1, \dots, T_N\}$ as

$$N(t) = \frac{1}{P(T_0, T_1)} \cdots \frac{1}{P(T_{i-1}, T_i)} \frac{P(t, T_{i+1})}{P(T_i, T_{i+1})}$$

for $T_i < t \leq T_{i+1}$.

For *discrete* model the short bond $P(t, T_{i+1})$ is not defined

→ make it *delayed*:

the bond $P(t, T_{i+1})$ depends of the "life" before T_i ,

so its diffusion term equals to zero for $t \in (T_i, T_{i+1})$

The RS numeraire is free of diffusion term:

$$dN(t) = N(t) \frac{\partial \log P(t, T_{i+1})}{\partial t} dt \text{ for } T_i < t \leq T_{i+1}$$

Remark.

Schlögl (1999) established equivalence of the RS measure and the risk-neutral one for the delayed (with zero diffusion term) short bond $P(t, T_{i+1})$.

LMM evolution in the RS measure.

The RS numeraire

$$N(t) = \frac{1}{P(T_0, T_1)} \cdots \frac{1}{P(T_{i-1}, T_i)} \frac{P(t, T_{i+1})}{P(T_i, T_{i+1})}$$

for $T_i \leq t \leq T_{i+1}$.

Restore the drifts for the Libors

$$\begin{aligned} dL_n(t) &= (L_n(t) b_n + L_n(0) (1 - b_n)) \\ &\times \gamma_n(t) \cdot \left(\sum_{j=\eta(t)}^n \alpha_j(t) dt + dW(t) \right) \end{aligned}$$

where tenor index

$$\eta(t) = n + 1 \quad \text{for } T_n < t \leq T_{n+1}$$

CC LMM setup.

- Domestic, foreign and FX models are driven with F independent Brownian motions $W_f(t)$ for $f = 1, \dots, F$
- Both domestic and foreign LMM's share the same time tenor T_i . This technical assumption dramatically simplifies the life.
- The domestic model volatility vectors

$$\gamma_n(t) = \{\gamma_{n,1}(t) \cdots \gamma_{n,F}(t)\}, n = 0, \dots, N - 1$$
- The foreign model volatility vectors

$$\tilde{\gamma}_n(t) = \{\tilde{\gamma}_{n,1}(t) \cdots \tilde{\gamma}_{n,F}(t)\}, n = 0, \dots, N - 1$$
- The FX-rate volatility vector

$$\sigma_X(t) = \{\sigma_{X1}(t) \cdots, \sigma_{XF}(t)\}$$
- Domestic and foreign shifts: b_n and \tilde{b}_n for $n = 0, \dots, N - 1$

CC LMM SDE's.

Choose the CC model measure as *domestic RS* \rightarrow

$$\begin{aligned}
 dL_n(t) &= (L_n(t) b_n + L_n(0) (1 - b_n)) \\
 &\quad \times \gamma_n(t) \cdot \left(\sum_{j=\eta(t)}^n \alpha_j(t) dt + dW(t) \right) \\
 d\tilde{L}_n(t) &= (\tilde{L}_n(t) \tilde{b}_n + \tilde{L}_n(0) (1 - \tilde{b}_n)) \\
 &\quad \times \tilde{\gamma}_n(t) \cdot \left(\sum_{j=\eta(t)}^n \tilde{\alpha}_j(t) dt - \sigma_X dt + dW(t) \right) \\
 dY(t) &= Y(t) \sigma_X(t) \cdot dW(t)
 \end{aligned}$$

where $Y(t)$ is the RS-discounted FX-rate

$$Y(t) = \frac{X(t) \tilde{N}(t)}{N(t)}$$

Remarks.

- The FX-rate has a log-normal diffusion term

$$dX(t) = \dots + X(t) \sigma_X(t) \cdot dW(t)$$

due to zero diffusion in the numeraires.

- The extra drift term in the foreign Libors evolution $-\sigma_X dt$ is due to a measure change (the foreign RS \rightarrow the domestic RS) established by the log-normal process $Y(t)$

$$d\tilde{W} = dW - \sigma_X dt$$

CC model calibration.

Standard approach to CC model calibration

- fix the common tenor T_i
- calibrate separately interest rate LMM vols to its own set of options or correlations
- fix correlation structure
- calibrate FX-vol to a set of FX-options given IR vols

The FX-calibration

→ requires analytical approximation of FX-option.

FX-option analytical approximation.

FX-option exercise falls exactly on a tenor date T_M

$$E \left[\frac{(X(T_M) - K)^+}{N(T_M)} \right]$$

(average is taken over domestic rolling-spot measure).

Proceed in the standard way:

- change measure to domestic forward one associated with bond $P(t, T_M)$ as numeraire.
- introduce associated forward FX-rate

$$F(t, T_M) = \frac{X(t) \tilde{P}(t, T_M)}{P(t, T_M)}$$

- rewrite the FX-option price in the forward measure in terms of the forward FX-rate

FX-option via forward FX-rate.

$$\begin{aligned} E \left[\frac{(X(T_M) - K)^+}{N(T_M)} \right] &= P(0, T_M) E_{T_M} \left[(X(T_M) - K)^+ \right] \\ &= P(0, T_M) E_{T_M} \left[(F(T_M, T_M) - K)^+ \right] \end{aligned}$$

The only process to follow is the forward FX-rate in the forward measure.

Simplify notations:

- all averages below are taken in the forward measure and denoted as $E[\dots]$
- the forward FX-rate is denoted as $F(t) \equiv F(t, T_M)$

Forward FX-rate.

Forward FX-rate is a martingale in the forward measure

→ calculate its diffusion term. For this:

– rewrite $F(t)$ in terms of RS-discounted FX-rate

$$F(t) = \frac{X(t) \tilde{P}(t, T_M)}{P(t, T_M)} = \frac{Y(t) \frac{N(t)}{P(t, T_M)}}{\frac{\tilde{N}(t)}{\tilde{P}(t, T_M)}}$$

– express bonds $P(t, T_M)$ via the bond ratios $R_n(t) = \frac{P(t, T_n)}{P(t, T_{n+1})}$

$$P(t, T_M) = P(t, T_{\eta(t)}) \prod_{n=\eta(t)}^{M-1} R_n^{-1}(t)$$

$$\Rightarrow \frac{N(t)}{P(t, T_M)} = \prod_{k=0}^{\eta(t)-1} \frac{1}{P(T_k, T_{k+1})} \prod_{n=\eta(t)}^{M-1} R_n(t)$$

$$F(t) = \prod_{k=0}^{\eta(t)-1} \frac{\tilde{P}(T_k, T_{k+1})}{P(T_k, T_{k+1})} Y(t) \prod_{n=\eta(t)}^{M-1} R_n(t) \tilde{R}_n^{-1}(t)$$

where bond ratios $R_n(t)$, $\tilde{R}_n(t)$ follow displaced diffusion processes and RS-discounted FX-rate $Y(t)$ has the log-normal diffusion part.

To proceed, rescale the forward rate and bond ratios to have unit initial values

$$\frac{F(t)}{F(0)} \rightarrow F(t)$$

and

$$\frac{R_n(t)}{R_n(0)} \rightarrow R_n(t) \text{ and } \frac{\tilde{R}_n(t)}{\tilde{R}_n(0)} \rightarrow \tilde{R}_n(t) \text{ for any } n$$

Forward FX-rate SDE.

Applying Ito's lemma and using SDE's of bond ratios and the RS-discounted FX-rate one has

$$dF(t) = \Sigma(t) \cdot dW(t)$$

(the process is martingale). The explicit expression

$$\Sigma(t) = F(t) \Lambda(t)$$

where

$$\begin{aligned} \Lambda(t) &= \sigma_X(t) \\ &+ \sum_{n=\eta(t)}^{M-1} \frac{\beta_n R_n(t) + 1 - \beta_n}{R_n(t)} \sigma_n(t) - \frac{\tilde{\beta}_n \tilde{R}_n(t) + 1 - \tilde{\beta}_n}{\tilde{R}_n(t)} \tilde{\sigma}_n(t) \end{aligned}$$

Obviously, the process $F(t)$ is non-Markovian.

Optimal time-dependent displaced diffusion coefficients.

–optimal time-dependent volatility

$$\sigma(t) = \sigma_X(t) + \sum_{n=\eta(t)}^{M-1} \sigma_n(t) - \tilde{\sigma}_n(t) + O(\sigma_+^3)$$

–optimal time-dependent shift

$$1 - \beta(t) = \frac{\sum_{n=\eta(t)}^{M-1} (1 - \beta_n) D_n(t) \int_0^t D_n(u) du}{|\sigma(t)|^2 \int_0^t |\sigma(u)|^2 du} - \frac{\sum_{n=\eta(t)}^{M-1} (1 - \tilde{\beta}_n) \tilde{D}_n(t) \int_0^t \tilde{D}_n(u) du}{|\sigma(t)|^2 \int_0^t |\sigma(u)|^2 du} + O(\sigma_+^2)$$

where $D_n(t) = \sigma_n(t) \cdot \sigma(t)$ and $\tilde{D}_n(t) = \tilde{\sigma}_n(t) \cdot \sigma(t)$

FX-option implied vol: analytics vs. simulations

Before CC LMM data setup \rightarrow a word on correlations.

We use 3F setup (all volatility vectors have size 3) but consider individual models degenerate to 1F.

Domestic model volatility vectors

$$\gamma_n(t) = \lambda_n(t) \mathcal{N}$$

where $\lambda_n(t)$ is a scalar and \mathcal{N} is flat and common for all libors direction vector $|\mathcal{N}| = 1$.

The same for the foreign component

$$\tilde{\gamma}_n(t) = \tilde{\lambda}_n(t) \tilde{\mathcal{N}}$$

and FX-rate volatility

$$\sigma_X(t) = \lambda_X(t) \mathcal{N}_X$$

One can associate correlated scalar Brownian motions to the three components

- Domestic

$$dZ(t) = \mathcal{N} \cdot dW(t) \Rightarrow \gamma_n(t) \cdot dW(t) = \lambda_n(t) dZ(t)$$

- Foreign

$$d\tilde{Z}(t) = \tilde{\mathcal{N}} \cdot dW(t) \Rightarrow \tilde{\gamma}_n(t) \cdot dW(t) = \tilde{\lambda}_n(t) d\tilde{Z}(t)$$

- FX-rate

$$dZ_X(t) = \mathcal{N}_X \cdot dW(t) \Rightarrow \sigma_X(t) \cdot dW(t) = \lambda_X(t) dZ_X(t)$$

Correlations between different markets are dot products of the corresponding directions or correlations between underlying effective Brownian motions $Z(t)$, $\tilde{Z}(t)$, $Z_X(t)$.

CC LMM data

- Typical (before the crisis) yield/volatility data for EUR(=domestic) and USD(=foreign)
- Regular SA Libor dates $T_i = 0.5Y(1 + i)$ up to 30Y
- 1F (non-shifted) BGM models for both IR markets
- Flat domestic and foreign LMM volatilities calibrated to the respective ATM caplets
- FX spot is 100 (not really realistic but good for visualization!)
- Correlations: domestic-foreign 0.25, domestic-FX -0.15 and foreign-FX -0.20

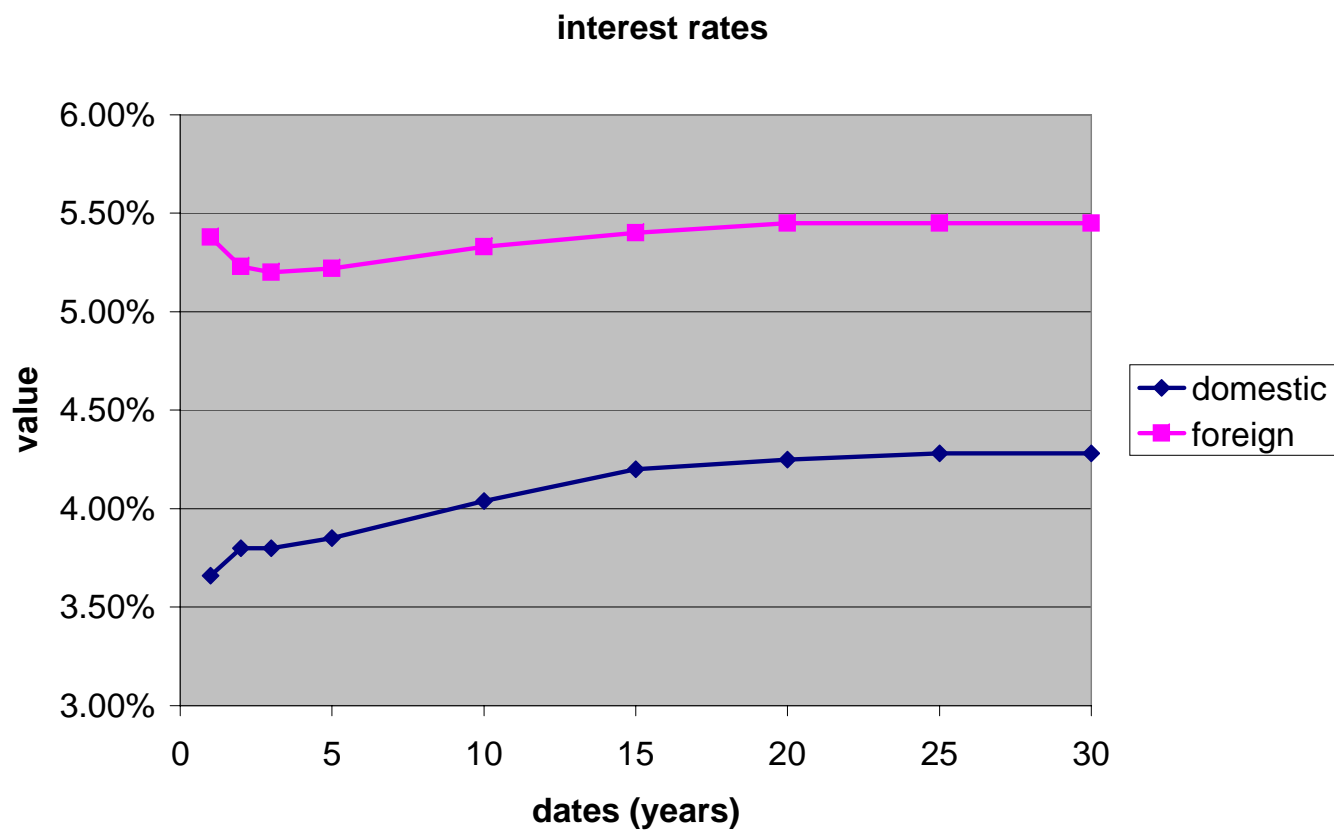


Figure 3: Interest rate curves

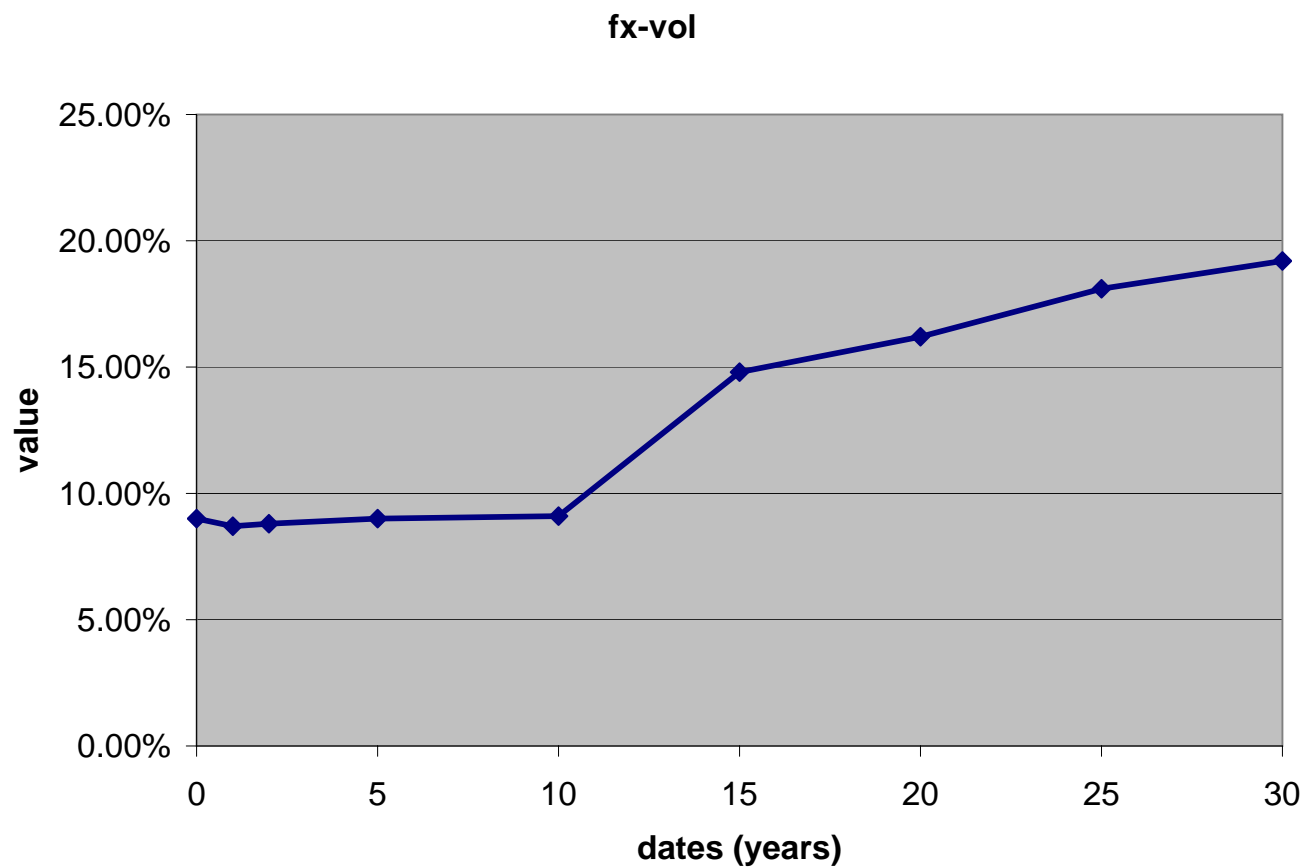


Figure 4: Instantaneous FX-rate (scalar) volatility $\lambda_X(t)$

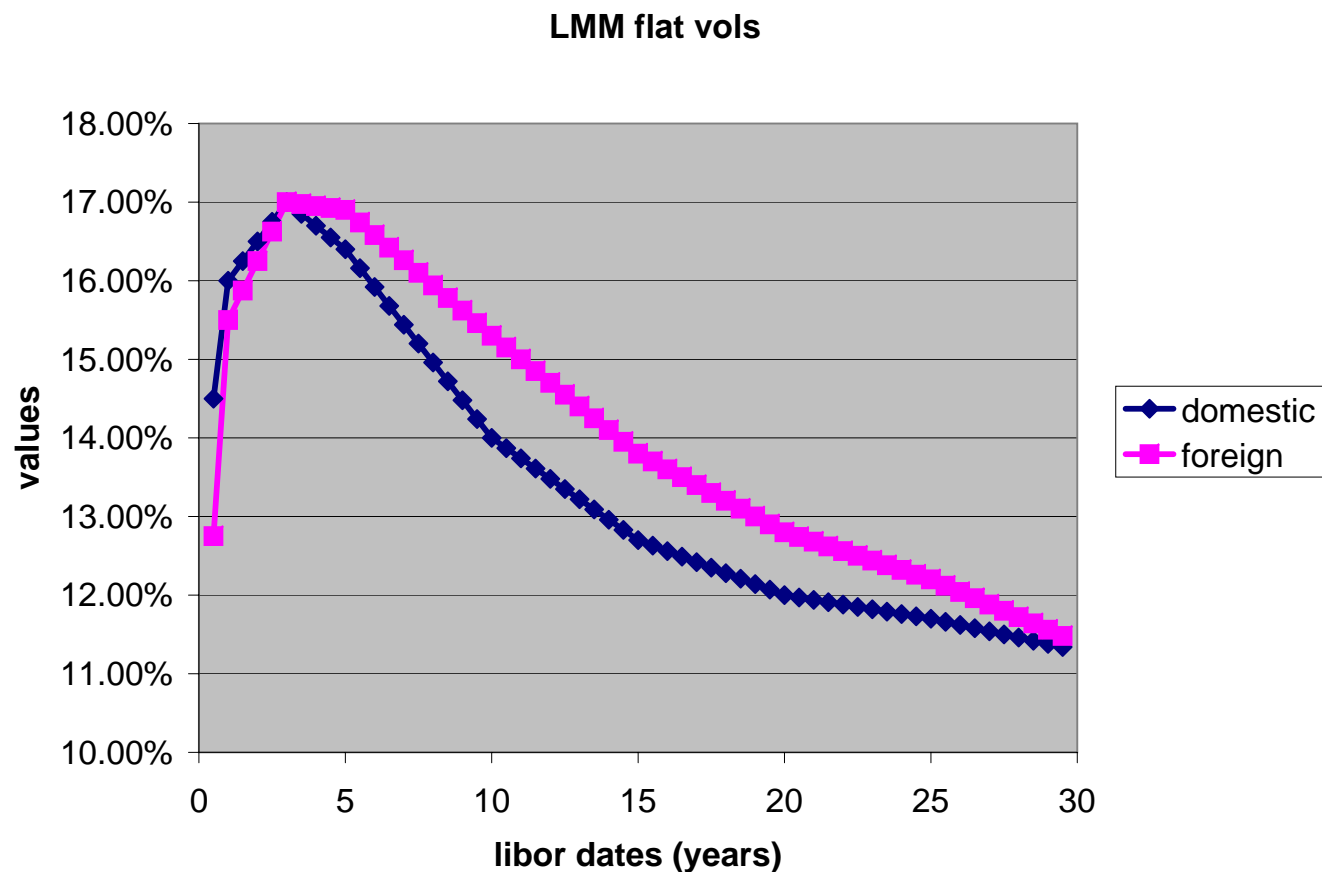


Figure 5: LMM flat (scalar) volatilities for domestic and foreign markets, λ_n and $\tilde{\lambda}_n$, as functions of T_n

FX-option exercises and strikes

–Exercises $t_i = \{5Y, 10Y, 15Y, 20Y, 25Y, 30Y\}$

–Large range of strikes (exercise dependent)

exer	1	2	3	4	5	6	7
5Y	66.77	74.67	83.50	93.38	104.43	116.78	130.59
10Y	54.70	64.07	75.04	87.90	102.95	120.59	141.25
15Y	46.72	56.71	68.82	83.53	101.37	123.04	149.32
20Y	40.22	50.30	62.90	78.66	98.37	123.03	153.85
25Y	35.26	45.27	58.13	74.64	95.84	123.06	158.01
30Y	30.96	40.71	53.53	70.40	92.58	121.74	160.09

Strikes in **bold** (# 4) are ATM.

FX-option implied vols.

Table of FX-option implied vols (%) for all strikes and exercises calculated with the MP approximation

exer	1	2	3	4	5	6	7
5Y	9.43	9.40	9.38	9.36	9.34	9.32	9.31
10Y	11.03	10.90	10.78	10.68	10.58	10.50	10.42
15Y	13.65	13.43	13.24	13.07	12.92	12.78	12.66
20Y	16.47	16.19	15.95	15.74	15.55	15.39	15.26
25Y	18.95	18.60	18.31	18.07	17.85	17.67	17.52
30Y	21.25	20.84	20.50	20.21	19.97	19.77	19.60

FX-option pricing errors.

Table of errors in FX-option implied vols (*bps*) for all strikes and exercises between the MP approximation and simulation

exer	1	2	3	4	5	6	7
5Y	-1	-1	-1	-1	-1	-1	-1
10Y	-8	-6	-5	-4	-5	-6	-7
15Y	-10	-8	-6	-6	-7	-9	-13
20Y	-7	-7	-6	-7	-9	-11	-15
25Y	-1	-2	-4	-6	-9	-12	-16
30Y	11	5	1	-4	-7	-10	-15

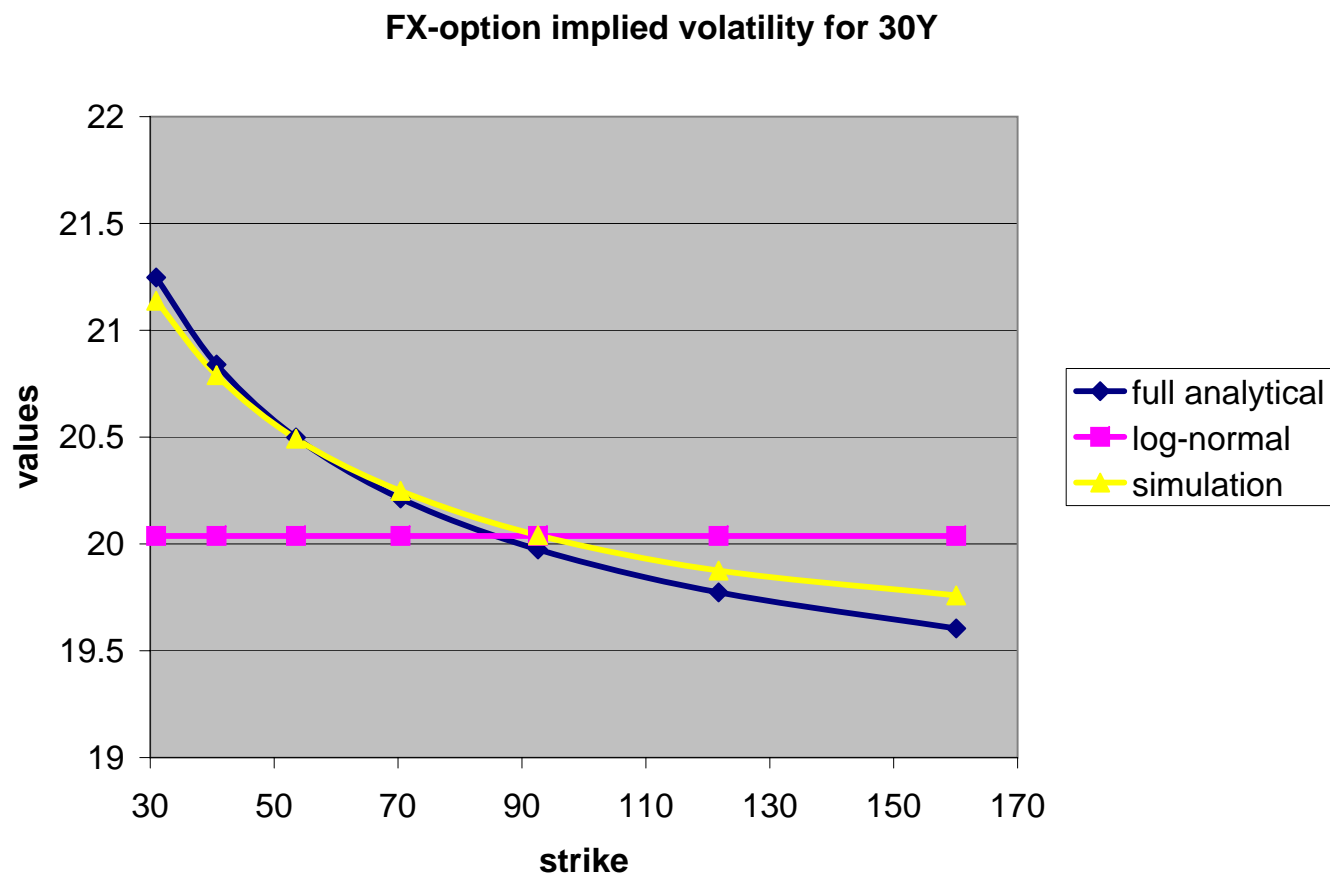


Figure 6: Comparison of the full (optimally skewed) analytical solution, the log-normal approximation and simulation (30Y).

Observations/remarks

- Good approximation quality for the optimal analytics: the biggest error does not exceed 7 bps (for ATM) and 16 bps (for out-of-money).
- The FX-option skew is "induced" by IR's.
- Absence of a free parameter for skew control
→ calibration only to ATM FX-options.

Summary

- Advantages
 - Straightforward and universal method
 - Volatility skew is captured
 - Good accuracy for typical cases
- Drawbacks
 - Volatility smile (convexity) is not matched (except for the LMM SV case)
 - Accuracy gets worse for highly heterogeneous cases

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