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MARKET MODELS OF LIBORS AND SWAP RATES

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Contents

Suggested readings:

- Marek Musiela and Marek Rutkowski: *Martingale Methods in Financial Modelling.* 2nd edition. Springer-Verlag, Berlin Heidelberg New York, 2005.
- Damiano Brigo and Fabio Mercurio: *Interest Rate Models: Theory and Practice.* Springer-Verlag, Berlin Heidelberg New York, 2001.
- Phil Hunt and Joanne Kennedy: *Financial Derivatives in Theory and Practice.* J. Wiley & Sons, Chichester New York, 2000.
- Jessica James and Nick Webber: *Interest Rate Modelling.* J. Wiley & Sons, Chichester New York, 2000.

1 Introduction

The last two decades were marked by a rapidly growing interest in the arbitrage-free modelling of bond market. Undoubtedly, one of the major achievements in this area was a new approach to the term structure modelling proposed by Heath, Jarrow and Morton in their work published in 1992, commonly known as the HJM methodology. One of its main features is that it covers a large variety of previously proposed models and provides a unified approach to the modelling of instantaneous interest rates and to the valuation of interest-rate sensitive derivatives. Let us first give a very concise description of the HJM approach (a more detailed account can be found, for instance, in Chapter 11 of the monograph by Musiela and Rutkowski (2005)).

1.1 Heath-Jarrow-Morton Approach

The HJM methodology is based on an exogenous specification of the dynamics of instantaneous, continuously compounded forward rates $f(t,T)$. For any fixed maturity $T \leq T^*$, the dynamics of the forward rate $f(t,T)$ are

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\cdot dW_t,$$

where α and σ are adapted stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively, and W is a d-dimensional standard Brownian motion with respect to the underlying probability measure $\mathbb{P}$ which plays the role of the real-world probability. More precisely, for every fixed $T \leq T^*$, where $T^* > 0$ is the horizon date, we have

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\cdot dW_u$$

for some Borel-measurable function $f(0,\cdot) : [0,T^*] \to \mathbb{R}$ and stochastic processes $\alpha(\cdot,T)$ and $\sigma(\cdot,T)$. Let us notice that, for any fixed maturity date $T \leq T^*$, the initial condition $f(0,T)$ is determined by the current value of the continuously compounded forward rate for the future date T which prevails at time 0. In practical terms, the function $f(0,T)$ is determined by the current yield curve, which can be estimated on the basis of observed market prices of bonds and/or other relevant instruments.

Let us denote by $B(t,T)$ the price at time $t \leq T$ of a unit zero-coupon bond maturing at time $T \leq T^*$. Formally, the price $B(t,T)$ can be recovered from the formula

$$B(t,T) = \exp\Big(-\int_t^T f(t,u)\,du\Big).$$

The problem of the absence of arbitrage opportunities in the bond market can be formulated in terms of the existence of a suitably defined martingale measure. It appears that in an arbitrage-free setting – that is, under a martingale measure – the drift coefficient α in dynamics of the instantaneous forward rate is uniquely determined by the volatility coefficient σ and an auxiliary stochastic process, which can be interpreted as the market price for interest-rate risk.

If we denote by $\mathbb{P}^*$ the (spot) martingale measure for the bond market and by W^* the associated standard Brownian motion, then we obtain

$$dB(t,T) = B(t,T)\big(r_t\,dt + b(t,T)\cdot dW_t^*\big),$$

where $r_t = f(t,t)$ is the short-term interest rate, and the bond price volatility $b(t,T)$ is given by the expression

$$b(t,T) = -\int_t^T \sigma(t,u)\,du. \tag{1}$$

In the special case when the coefficient σ follows a deterministic function, the valuation formulae for interest rate-sensitive derivatives are independent of the choice of the risk premium.

The HJM methodology appeared to be very successful both from the theoretical and practical viewpoints. However, since the HJM approach to the term structure modelling is based on an arbitrage-free dynamics of the instantaneous continuously compounded forward rates, it requires a certain degree of smoothness with respect to the tenor of the bond prices and their volatilities. For this reason, working with such models is not always convenient.

1.2 Modelling of Market Rates

An alternative construction of an arbitrage-free family of bond prices, making no reference to instantaneous rates, is in some circumstances more suitable. The first step in this direction was done by Sandmann and Sondermann (1993), who focused on the effective annual interest rate. This approach was further developed in ground-breaking papers by Miltersen et al. (1997) and Brace et al. (1997), who proposed to focus on a direct modelling of forward LIBORs. The main goal was to produce an arbitrage-free term structure model that would support the common practice of pricing typical interest-rate derivatives, such as caps and swaptions, through a suitable version of Black's formula. This practical requirement enforces the lognormality of the forward LIBOR (or swap) rate under the corresponding forward martingale measure.

Let us recall that, by market convention, the forward LIBOR over the future accrual period $[T, T+\delta]$, as seen at time t, is set to satisfy

$$1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)},$$

or equivalently,

$$L(t,T) = \frac{B(t,T) - B(t,T+\delta)}{B(t,T+\delta)}.$$

The last formula makes it obvious that the volatility of the forward LIBOR is not deterministic if the bond price volatility follows a deterministic function. For this reason the Black formula for caplets is manifestly incompatible with the Gaussian HJM model – that is, the HJM model in which the bond price volatility $b(t,T)$ is deterministic. Consequently, the "market formula" for caps cannot be derived in this setup (though the value of a caplet is given by a closed-form expression in the Gaussian HJM framework).

On the other hand, it is interesting to notice that Brace et al. (1997) parametrize their version of the lognormal LIBOR model introduced by Miltersen et al. (1997) with a piecewise constant volatility function. They need to consider smooth volatility functions in order to analyze the model in the HJM framework, however. The backward induction approach to the modelling of forward LIBOR and swap rate developed in Musiela and Rutkowski (1997b) and Jamshidian (1997) overcomes this technical difficulty. In contrast to the previous papers, it allows also for the modelling of forward LIBORs (and forward swap rates) associated with accrual periods of differing lengths.

A similar, but not identical, approach to the modelling of market rate was developed in a series of papers by Hunt et al. (1996, 2000) and Hunt and Kennedy (1997, 1998). Since emphasis is put here on the existence of the underlying low-dimensional Markov process that governs directly the dynamics of interest rates, this alternative approach is termed the *Markov-functional approach.* This property leads to a considerable simplification in numerical procedures associated with the model's implementation. An important feature of this approach is its ability of providing a perfect fit to market prices of a given family of interest-rate options (e.g., a family of digital swaptions with varying strikes). Another tractable term structure model is the *rational lognormal model* proposed by Flesaker and Hughston (1996a, 1996b) (see also Rutkowski (1997) and Jin and Glasserman (2001) in this regard).

Let mention that we use throughout the notation adopted in Musiela and Rutkowski (2005). The interested reader is referred to this monograph for more details on term structure modelling, as well as for the general background on arbitrage pricing theory.

2 Modelling of LIBORs

In this section, we present various approaches to the modelling of forward LIBORs. Due to the limited space, we focus on a model construction and its basic properties, as well as the valuation of the most typical derivatives. For further details and more recent developments, the interested reader is referred to the following papers: Musiela and Sondermann (1993), Sandmann and Sondermann (1993), Goldys et al. (1994), Sandmann et al. (1995), Brace et al. (1997), Jamshidian (1997, 1999), Miltersen et al. (1997), Musiela and Rutkowski (1997b), Rady (1997), Sandmann and Sondermann (1997), Rutkowski (1998, 1999), Yasuoka (1998), Jamshidian (1999), Andersen and Brotherton-Ratcliffe (2001), Jin and Glasserman (2001), Mikkelsen (2002), Schlögl (2002), Glasserman and Kou (2003), Joshi and Rebonato (2003), Piterbarg (2003a), and Pietersz and Regenmortel (2004).

Issues related to model's implementations (in particular, a model calibration and an arbitrage-free discretization of the lognormal model of forward LIBORs) are extensively treated in Brace (1996), Brace et al. (1998), Schlögl (1999), Yasuoka (1999), Lotz and Schlögl (1999), Rebonato (1999a, 1999b), Schoenmakers and Coffey (1999), Andersen (2000), Andersen and Andreasen (2000), Brace and Womersley (2000), Dun et al. (2000), Glasserman and Zhao (2000), Hull and White (2000), Sidenius (2000), De Jong et al. (2001a, 2001b), Kawai (2001), Longstaff et al. (2001), De Malherbe (2002), Joshi and Theis (2002), Wu (2002), d'Aspremont (2003), Brigo and Mercurio (2003), Glasserman and Merener (2003), Pelsser and Pietersz (2003), Piterbarg (2003b, 2003c), Galluccio et al. (2004), and Pelsser et al. (2004).

2.1 Forward and Futures LIBORs

Our first task is to examine these properties of forward and futures contracts related to the notion of the LIBOR which are universal; that is, which do not rely on specific assumptions imposed on a particular model of the term structure of interest rates. To this end, we fix an index j, and we consider various interest-rate sensitive derivatives related to the period $[T_j, T_{j+1}]$. To be more specific, we shall focus in this section on single-period forward swaps – that is, forward rate agreements.

We need to introduce some notation. We assume that we are given a prespecified collection of reset/settlement dates $0 < T_0 < T_1 < \cdots < T_n = T^*$, referred to as the *tenor structure.* Also, we denote $\delta_j = T_j - T_{j-1}$ for $j = 1, 2, \ldots, n$. We write $B(t, T_j)$ to denote the price at time t of a T_j-maturity zero-coupon bond. $\mathbb{P}^*$ is the *spot martingale measure,* while for any $j = 0, 1, \ldots, n$ we write $\mathbb{P}_{T_j}$ to denote the *forward martingale measure* associated with the date T_j. The corresponding d-dimensional Brownian motions are denoted by W^* and W^{T_j}, respectively. Also, we write $F_B(t, T, U) = B(t, T)/B(t, U)$ so that

$$F_B(t, T_{j+1}, T_j) = \frac{B(t, T_{j+1})}{B(t, T_j)}, \quad \forall\, t \in [0, T_j],$$

is the forward price at time t of the T_{j+1}-maturity zero-coupon bond for the settlement date T_j. We use the symbol $\pi_t(X)$ to denote the value (i.e., the arbitrage price) at time t of a European contingent claim X. Finally, we shall use the letter $\mathcal{E}$ for the Doléans exponential, for instance,

$$\mathcal{E}_t\Big(\int_0^{\cdot} \gamma_u \cdot dW_u^*\Big) = \exp\Big(\int_0^t \gamma_u \cdot dW_u^* - \frac{1}{2}\int_0^t |\gamma_u|^2\, du\Big),$$

where the dot '$\cdot$' and $|\cdot|$ stand for the inner product and Euclidean norm in $\mathbb{R}^d$, respectively.

2.1.1 Single-period Swaps Settled in Arrears

Let us first consider a single-period swap agreement settled in arrears; i.e., with the *reset date* T_j and the *settlement date* T_{j+1} (multi-period interest rate swaps are examined in Section 3). By the contractual features, the long party pays $\delta_{j+1}\kappa$ and receives $B^{-1}(T_j, T_{j+1}) - 1$ at time T_{j+1}. Equivalently, he pays an amount $Y_1 = 1 + \delta_{j+1}\kappa$ and receives $Y_2 = B^{-1}(T_j, T_{j+1})$ at this date.

The values at time $t \le T_j$ of these payoffs are

$$\pi_t(Y_1) = B(t, T_{j+1})\big(1 + \delta_{j+1}\kappa\big), \quad \pi_t(Y_2) = B(t, T_j).$$

The second equality above is trivial, since the payoff Y_2 is equivalent to the unit payoff at time T_j. Consequently, for any fixed $t \le T_j$, the value of the *forward swap rate,* which makes the contract valueless at time t, can be found by solving for $\kappa = \kappa(t, T_j, T_{j+1})$ the following equation

$$\pi_t(Y_2) - \pi_t(Y_1) = B(t, T_j) - B(t, T_{j+1})\big(1 + \delta_{j+1}\kappa\big) = 0.$$

It is thus apparent that

$$\kappa(t, T_j, T_{j+1}) = \frac{B(t, T_j) - B(t, T_{j+1})}{\delta_{j+1} B(t, T_{j+1})}, \quad \forall\, t \in [0, T_j].$$

Note that the forward swap rate $\kappa(t, T_j, T_{j+1})$ coincides with the *forward LIBOR* $L(t, T_j)$ which, by the market convention, is set to satisfy, for every $t \in [0, T_j]$,

$$1 + \delta_{j+1} L(t, T_j) = \frac{B(t, T_j)}{B(t, T_{j+1})} = \mathbb{E}_{\mathbb{P}_{T_{j+1}}}\big(B^{-1}(T_j, T_{j+1}) \,|\, \mathcal{F}_t\big). \tag{2}$$

Let us notice that the last equality is a consequence of the definition of the forward measure $\mathbb{P}_{T_{j+1}}$. We conclude that in order to determine the forward LIBOR $L(\cdot, T_j)$, it is enough to find the forward price $F_X(t, T_{j+1})$ at time t of the contingent claim $X = B^{-1}(T_j, T_{j+1})$ in the forward contact that settles at time T_{j+1}. Indeed, it is well known (see, for instance, Musiela and Rutkowski (2005)) that

$$F_X(t, T_{j+1}) = B(t, T_{j+1})\, \mathbb{E}_{\mathbb{P}_{T_{j+1}}}\big(B^{-1}(T_j, T_{j+1}) \,|\, \mathcal{F}_t\big).$$

Furthermore, it is evident that the process $L(\cdot, T_j)$ follows a martingale under the forward probability measure $\mathbb{P}_{T_{j+1}}$. Recall that in the Heath-Jarrow-Morton framework, we have, under $\mathbb{P}_{T_{j+1}}$,

$$dF_B(t, T_j, T_{j+1}) = F_B(t, T_j, T_{j+1})\big(b(t, T_j) - b(t, T_{j+1})\big) \cdot dW_t^{T_{j+1}}, \tag{3}$$

where for each maturity date T the process $b(\cdot, T)$ represents the price volatility of the T-maturity zero-coupon bond. Furthermore, if the process $L(\cdot, T_j)$ is strictly positive, it can be shown to admit the following representation[1]

$$dL(t, T_j) = L(t, T_j)\lambda(t, T_j) \cdot dW_t^{T_{j+1}},$$

where $\lambda(\cdot, T_j)$ is an adapted stochastic process which satisfies mild integrability conditions. Combining the last two formulae with (2), we arrive at the following fundamental relationship, which plays an essential role in the construction of the lognormal model of forward LIBORs,

$$\frac{\delta_{j+1} L(t, T_j)}{1 + \delta_{j+1} L(t, T_j)}\, \lambda(t, T_j) = b(t, T_j) - b(t, T_{j+1}), \quad \forall\, t \in [0, T_j]. \tag{4}$$

For instance, in the construction which is based on the backward induction, relationship (4) will allow us to determine the forward measure for the date T_j, provided that $\mathbb{P}_{T_{j+1}}$, $W^{T_{j+1}}$ and the volatility $\lambda(t, T_j)$ of the forward LIBOR $L(\cdot, T_{j-1})$ are known. One may assume, for instance, that $\lambda(\cdot, T_j) : [0, T_j] \to \mathbb{R}^d$ is a prespecified deterministic function. Recall also that in the Heath-Jarrow-Morton framework[2] the Radon-Nikodým density of $\mathbb{P}_{T_j}$ with respect to $\mathbb{P}_{T_{j+1}}$ is known to satisfy

$$\frac{d\mathbb{P}_{T_j}}{d\mathbb{P}_{T_{j+1}}} = \mathcal{E}_{T_j}\Big(\int_0^{\cdot} \big(b(t, T_j) - b(t, T_{j+1})\big) \cdot dW_t^{T_{j+1}}\Big). \tag{5}$$

[1]This representation is a consequence of the martingale representation property of the standard Brownian motion.
[2]See Heath et al. (1992).

In view of (4), we thus have

$$\frac{d\mathbb{P}_{T_j}}{d\mathbb{P}_{T_{j+1}}} = \mathcal{E}_{T_j}\Big(\int_0^{\cdot} \frac{\delta_{j+1}L(t,T_j)}{1+\delta_{j+1}L(t,T_j)}\,\lambda(t,T_j)\cdot dW_t^{T_{j+1}}\Big).$$

For our further purposes, it is also useful to observe that this density admits the following representation

$$\frac{d\mathbb{P}_{T_j}}{d\mathbb{P}_{T_{j+1}}} = cF_B(T_j,T_j,T_{j+1}) = c\big(1+\delta_{j+1}L(T_j,T_j)\big), \quad \mathbb{P}_{T_{j+1}}\text{-a.s.}, \tag{6}$$

where $c > 0$ is the normalizing constant, and thus

$$\frac{d\mathbb{P}_{T_j}}{d\mathbb{P}_{T_{j+1}}}_{\,|\mathcal{F}_t} = cF_B(t,T_j,T_{j+1}) = c\big(1+\delta_{j+1}L(t,T_j)\big), \quad \mathbb{P}_{T_{j+1}}\text{-a.s.}$$

Finally, the dynamics of the process $L(\cdot,T_j)$ under the probability measure $\mathbb{P}_{T_j}$ are given by the following stochastic differential equation:

$$dL(t,T_j) = L(t,T_j)\left(\frac{\delta_{j+1}L(t,T_j)|\lambda(t,T_j)|^2}{1+\delta_{j+1}L(t,T_j)}\,dt + \lambda(t,T_j)\cdot dW_t^{T_j}\right).$$

As we shall see in what follows, it is nevertheless not hard to determine the probability law of $L(\cdot,T_j)$ under the forward measure $\mathbb{P}_{T_j}$ – at least in the case of the deterministic volatility $\lambda(\cdot,T_j)$ of the forward LIBOR.

2.1.2 Single-period Swaps Settled in Advance

We have assumed that the LIBOR is fixed at the beginning of the interest accrual period, and paid at the end. Consider now a swap which is settled in advance – that is, at time T_j. Our first goal is to determine the forward swap rate implied by such a contract. Note that under the present assumptions, the long party (formally) pays an amount $Y_1 = 1+\delta_{j+1}\kappa$ and receives $Y_2 = B^{-1}(T_j,T_{j+1})$ at the settlement date T_j (which coincides here with the reset date). The values at time $t \le T_j$ of these payoffs admit the following representations

$$\pi_t(Y_1) = B(t,T_j)\big(1+\delta_{j+1}\kappa\big), \quad \pi_t(Y_2) = B(t,T_j)\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j,T_{j+1})\,|\,\mathcal{F}_t).$$

The value $\kappa = \widehat{\kappa}(t,T_j,T_{j+1})$ of the modified forward swap rate, which makes the swap agreement settled in advance valueless at time t, can be found from the equality

$$\pi_t(Y_2) - \pi_t(Y_1) = B(t,T_j)\big(\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j,T_{j+1})\,|\,\mathcal{F}_t) - (1+\delta_{j+1}\kappa)\big) = 0.$$

It is clear that

$$\widehat{\kappa}(t,T_j,T_{j+1}) = \delta_{j+1}^{-1}\big(\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j,T_{j+1})\,|\,\mathcal{F}_t) - 1\big).$$

We are in a position to introduce the *modified forward LIBOR* $\widetilde{L}(t,T_j)$ by setting, for every $t \in [0,T_j]$,

$$\widetilde{L}(t,T_j) := \delta_{j+1}^{-1}\big(\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j,T_{j+1})\,|\,\mathcal{F}_t) - 1\big).$$

The modified forward LIBOR is also known as *LIBOR in arrears* (since the rate is fixed just before it is paid).

Let us make two remarks. First, finding of the modified forward LIBOR $\widetilde{L}(\cdot,T_j)$ is formally equivalent to finding the forward price of the claim $B^{-1}(T_j,T_{j+1})$ for the settlement date T_j.[3] Second, it is useful to observe that

$$\widetilde{L}(t,T_j) = \mathbb{E}_{\mathbb{P}_{T_j}}\Big(\frac{1-B(T_j,T_{j+1})}{\delta_{j+1}B(T_j,T_{j+1})}\,\Big|\,\mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{P}_{T_j}}(L(T_j,T_j)\,|\,\mathcal{F}_t). \tag{7}$$

[3] Recall that in the case of a forward LIBOR, the settlement date was T_{j+1}.

In particular, it is evident that at the reset date T_j the two kinds of forward LIBORs introduced above coincide, since manifestly

$$\widetilde{L}(T_j, T_j) = \frac{1 - B(T_j, T_{j+1})}{\delta_{j+1} B(T_j, T_{j+1})} = L(T_j, T_j).$$

To summarize, the 'standard' forward LIBOR $L(\cdot, T_j)$ satisfies

$$L(t, T_j) = \mathbb{E}_{\mathbb{P}_{T_{j+1}}}(L(T_j, T_j) \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T_j],$$

with the initial condition

$$L(0, T_j) = \frac{B(0, T_j) - B(0, T_{j+1})}{\delta_{j+1} B(0, T_{j+1})}.$$

On the other hand, for the modified LIBOR $\widetilde{L}(\cdot, T_j)$ we have

$$\widetilde{L}(t, T_j) = \mathbb{E}_{\mathbb{P}_{T_j}}(\widetilde{L}(T_j, T_j) \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T_j],$$

with the initial condition

$$\widetilde{L}(0, T_j) = \delta_{j+1}^{-1}\big(\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j, T_{j+1})) - 1\big).$$

The calculation of the right-hand side above involves not only the initial term structure, but also the volatilities of bond prices (for more details, we refer to Rutkowski (1998)).

2.1.3 Eurodollar Futures Contracts

The next object of our studies is the futures LIBOR. A *Eurodollar futures contract* is a futures contract in which the LIBOR plays the role of an underlying asset. By convention, at the contract's maturity date T_j, the quoted Eurodollar futures price, denoted by $E(T_j, T_j)$, is set to satisfy

$$E(T_j, T_j) := 1 - \delta_{j+1} L(T_j, T_j).$$

Equivalently, in terms of the zero-coupon bond price we have $E(T_j, T_j) = 2 - B^{-1}(T_j, T_{j+1})$. From the general theory, it follows that the Eurodollar futures price at time $t \leq T_j$ equals

$$E(t, T_j) := \mathbb{E}_{\mathbb{P}^*}(E(T_j, T_j)) = 2 - \mathbb{E}_{\mathbb{P}^*}\big(B^{-1}(T_j, T_{j+1}) \,|\, \mathcal{F}_t\big) \tag{8}$$

(recall that $\mathbb{P}^*$ represents the spot martingale measure in a given model of the term structure). It is thus natural to introduce the concept of the *futures LIBOR,* associated with the Eurodollar futures contract, through the following definition.

Definition 2.1 Let $E(t, T_j)$ be the Eurodollar futures price at time t for the settlement date T_j. The implied *futures LIBOR* $L^f(t, T_j)$ satisfies

$$E(t, T_j) = 1 - \delta_{j+1} L^f(t, T_j), \quad \forall\, t \in [0, T_j]. \tag{9}$$

It follows immediately from (8)–(9) that the following equality is valid

$$1 + \delta_{j+1} L^f(t, T_j) = \mathbb{E}_{\mathbb{P}^*}\big(B^{-1}(T_j, T_{j+1}) \,|\, \mathcal{F}_t\big). \tag{10}$$

Equivalently, we have

$$L^f(t, T_j) = \mathbb{E}_{\mathbb{P}^*}(L(T_j, T_j) \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(\widetilde{L}(T_j, T_j) \,|\, \mathcal{F}_t).$$

Note that in any term structure model, the futures LIBOR necessarily follows a martingale under the spot martingale measure $\mathbb{P}^*$ (provided, of course, that $\mathbb{P}^*$ is well-defined in this model).

2.2 Lognormal LIBOR Models

We shall now describe alternative approaches to the modelling of forward LIBOR rates in the continuous- and discrete-tenor setups.

2.2.1 Miltersen-Sandmann-Sondermann Approach

The first attempt to construct a lognormal model of forward LIBORs was done by Miltersen et al. (1997). The interested reader is referred also to Musiela and Sondermann (1993), Goldys et al. (1994), and Sandmann et al. (1995) for related previous studies. As a starting point in their approach, Miltersen et al. (1997) postulate that the forward LIBOR $L(\cdot, T)$ satisfies

$$dL(t,T) = \mu(t,T)\,dt + L(t,T)\lambda(t,T)\cdot dW_t^*$$

with a deterministic volatility function $\lambda(\cdot, T) : [0,T] \to \mathbb{R}^d$. It is not difficult to deduce from the last formula that the forward price $F_B(t, T+\delta, T) = B(t, T+\delta)/B(t,T)$ satisfies under the forward measure $\mathbb{P}_T$ the following SDE

$$dF_B(t, T+\delta, T) = -F_B(t, T+\delta, T)\big(1 - F_B(t, T+\delta, T)\big)\lambda(t,T)\cdot dW_t^T.$$

Miltersen et al. (1997) focused on the partial differential equation satisfied by the function $v = v(t,x)$, which expresses the forward price of the bond put option in terms of the forward bond price. The PDE for the option price is

$$\frac{\partial v}{\partial t} + \frac{1}{2}|\lambda(t,T)|^2 x^2 (1-x)^2 \frac{\partial^2 v}{\partial x^2} = 0 \tag{11}$$

with the terminal condition $v(T,x) = (K-x)^+$. It is interesting to note that the PDE (11) was previously solved by Rady and Sandmann (1994), who worked within a different framework, however.[4] Using this solution, Miltersen et al. (1997) obtained not only the closed-form expression for the price of a bond option (this was already achieved in Rady and Sandmann (1994)), but also the "market formula" for a caplet. A rigorous approach to the problem of existence of a model postulated by Miltersen et al. (1997) was subsequently developed by Brace et al. (1997), who also worked within the continuous-time Heath-Jarrow-Morton framework.

2.2.2 Brace-Gątarek-Musiela Approach

To formally introduce the notion of a *forward LIBOR,* we assume that we are given a family $B(t,T)$ of bond prices, and thus also the collection $F_B(t,T,U)$ of forward processes. In contrast to the previous section, we shall now assume that a strictly positive real number δ representing the length of the accrual period is fixed. By definition, the forward δ-LIBOR $L(t,T)$ for the future date $T \le T^* - \delta$ prevailing at time t is given by the conventional market formula

$$1 + \delta L(t,T) = F_B(t,T,T+\delta) = \frac{B(t,T)}{B(t,T+\delta)}, \quad \forall\, t \in [0,T]. \tag{12}$$

The forward LIBOR $L(t,T)$ represents the add-on rate prevailing at time t over the future time interval $[T, T+\delta]$. In particular, the initial term structure of forward LIBORs satisfies

$$L(0,T) = \delta^{-1}\Big(\frac{B(0,T)}{B(0,T+\delta)} - 1\Big). \tag{13}$$

It is not hard to derive the dynamics of the forward LIBOR within the HJM framework.

[4]In fact, they were concerned with the valuation of options on zero-coupon bonds for the term structure model put forward by Bühler and Käsler (1989).

Proposition 2.1 *Forward LIBOR $L(\cdot,T)$ satisfies, under the forward measure $\mathbb{P}_{T+\delta}$ for the date $T+\delta$,*

$$dL(t,T) = \delta^{-1} F_B(t,T,T+\delta)\,\gamma(t,T,T+\delta)\cdot dW_t^{T+\delta},$$

where $\gamma(t,T,T+\delta) = b(t,T) - b(t,T+\delta)$, and the Brownian motion $W^{T+\delta}$ equals

$$W_t^{T+\delta} = W_t^* - \int_0^t b(u,T+\delta)\,du, \quad \forall\, t \in [0,T+\delta].$$

Put another way, the process $L(\cdot,T)$ solves the equation

$$dL(t,T) = \delta^{-1}(1+\delta L(t,T))\,\gamma(t,T,T+\delta)\cdot dW_t^{T+\delta}, \tag{14}$$

subject to the initial condition (13). Suppose that forward LIBORs $L(t,T)$ are strictly positive. Then formula (14) can be rewritten as follows

$$dL(t,T) = L(t,T)\,\lambda(t,T)\cdot dW_t^{T+\delta}, \tag{15}$$

with the following relationship between the volatilities $\lambda(t,T)$ and $\gamma(t,T,T+\delta)$

$$\lambda(t,T) = \frac{1+\delta L(t,T)}{\delta L(t,T)}\,\gamma(t,T,T+\delta). \tag{16}$$

We conclude that $\lambda(t,T)$ is random if $\gamma(t,T,T+\delta)$ is deterministic (and vice versa).

The construction of a lognormal model of forward LIBORs relies on the following assumptions.

(LR.1) For any maturity $T \le T^* - \delta$, we are given a $\mathbb{R}^d$-valued, bounded deterministic function[5] $\lambda(\cdot,T)$, which represents the volatility of the forward LIBOR process $L(\cdot,T)$.

(LR.2) We are given a strictly decreasing and strictly positive initial term structure $B(0,T), T \in [0,T^*]$. The associated initial term structure $L(0,T)$ of forward LIBORs satisfies, for every $T \in [0,T*-\delta]$,

$$L(0,T) = \frac{B(0,T) - B(0,T+\delta)}{\delta B(0,T+\delta)}. \tag{17}$$

To produce a model satisfying (LR.1)–(LR.2), Brace et al. (1997) place themselves in the Heath-Jarrow-Morton setup and they assume that for every $T \in [0,T^*]$, the volatility $b(t,T)$ vanishes for every $t \in [(T-\delta)\vee 0, T]$. In essence, the construction elaborated in Brace et al. (1997) is based on the forward induction, as opposed to the backward induction, which we shall use in the next section. They start by postulating that the dynamics of $L(t,T)$ under the spot martingale measure $\mathbb{P}^*$ are governed by the following SDE

$$dL(t,T) = \mu(t,T)\,dt + L(t,T)\lambda(t,T)\cdot dW_t^*,$$

where λ is a deterministic function, and the drift coefficient μ is yet unspecified. Recall that the arbitrage-free dynamics of the instantaneous forward rate $f(t,T)$ are

$$df(t,T) = \sigma(t,T)\cdot\sigma^*(t,T)\,dt + \sigma(t,T)\cdot dW_t^*,$$

where $\sigma^*(t,T) = \int_t^T \sigma(t,u)\,du = -b(t,T)$. Using (16), we find that

$$\sigma^*(t,T+\delta) - \sigma^*(t,T) = \int_T^{T+\delta} \sigma(t,u)\,du = \frac{\delta L(t,T)}{1+\delta L(t,T)}\,\lambda(t,T).$$

To solve the last equation for σ^* in terms of L, it is necessary to impose some sort of initial condition on the coefficient σ.

[5]Volatility λ could well follow an adapted stochastic process; we deliberately focus here on a *lognormal LIBOR model* in which λ is deterministic.

Lemma 2.1 *Assume that $\sigma(t,T) = 0$ for $0 \le t \le T \le t+\delta$. The the following relationship is valid*

$$b(t,T) = -\sigma^*(t,T) = -\sum_{k=1}^{[\delta^{-1}(T-t)]} \frac{\delta L(t,T-k\delta)}{1+\delta L(t,T-k\delta)}\,\lambda(t,T-k\delta). \tag{18}$$

The existence and uniqueness of solutions to SDEs which govern the instantaneous forward rate $f(t,T)$ and the forward LIBOR $L(t,T)$ for σ^* given by (18) can be shown using forward induction. Taking this for granted, we conclude that $L(t,T)$ satisfies, under the spot martingale measure $\mathbb{P}^*$,

$$dL(t,T) = L(t,T)\sigma^*(t,T^*+\delta)\cdot\lambda(t,T)\,dt + L(t,T)\lambda(t,T)\cdot dW_t^*.$$

In this way, Brace et al. (1997) completely specified their model of forward LIBORs.

2.2.3 Musiela-Rutkowski Approach

In this section, we describe an alternative approach to the modelling of forward LIBORs; the construction presented below is based on Musiela and Rutkowski (1997b). Let us start by introducing some notation. We assume that we are given a prespecified collection of reset/settlement dates $0 < T_0 < T_1 < \cdots < T_n = T^*$, referred to as the tenor structure (by convention, $T_{-1} = 0$). Let us denote $\delta_j = T_j - T_{j-1}$ for $j = 0,1,\ldots,n$. Then obviously $T_j = \sum_{i=0}^{j}\delta_i$ for every $j = 0,1,\ldots,n$. We find it convenient to denote, for $m = 0,1,\ldots,n$,

$$T_m^* = T^* - \sum_{j=n-m+1}^{n}\delta_j = T_{n-m}.$$

For any $j = 0,1,\ldots,n-1$, we define the forward LIBOR $L(\cdot,T_j)$ by setting

$$L(t,T_j) = \frac{B(t,T_j) - B(t,T_{j+1})}{\delta_{j+1}B(t,T_{j+1})}, \quad \forall\, t \in [0,T_j].$$

Definition 2.2 For any $j = 0,1,\ldots,n,$ a probability measure $\mathbb{P}_{T_j}$ on $(\Omega,\mathcal{F}_{T_j})$, equivalent to $\mathbb{P}$, is said to be the *forward LIBOR measure* for the date T_j if, for every $k = 0,1,\ldots,n$ the relative bond price

$$U_{n-j+1}(t,T_k) := \frac{B(t,T_k)}{B(t,T_j)}, \quad \forall\, t \in [0,T_k \wedge T_j],$$

follows a local martingale under $\mathbb{P}_{T_j}$.

It is clear that the notion of forward LIBOR measure is in fact identical with that of a forward probability measure for a given date. Also, it is trivial to observe that the forward LIBOR $L(\cdot,T_j)$ necessarily follows a local martingale under the forward LIBOR measure for the date T_{j+1}. If, in addition, it is a strictly positive process, the existence of the associated volatility process can be justified by standard arguments.

In our further development, we shall go the other way around; that is, we will assume that for any date T_j, the volatility $\lambda(\cdot,T_j)$ of the forward LIBOR $L(\cdot,T_j)$ is exogenously given. In principle, it can be a deterministic $\mathbb{R}^d$-valued function of time, an $\mathbb{R}^d$-valued function of the underlying forward LIBOR rates, or it can follow a d-dimensional adapted stochastic process. For simplicity, we assume throughout that the volatilities of forward LIBORs are bounded processes (or functions). To be more specific, we make the following standing assumptions.

Assumptions (LR). We are given a set of bounded adapted processes $\lambda(\cdot,T_j), j = 0,1,\ldots,n-1$, representing the volatilities of forward LIBORs $L(\cdot,T_j)$. In addition, we are given an initial term structure of interest rates, specified by a family $B(0,T_j)$, $j = 0,1,\ldots,n$, of bond prices. We assume that $B(0,T_j) > B(0,T_{j+1})$ for $j = 0,1,\ldots,n-1$.

Our aim is to construct a family $L(\cdot,T_j),\ j=0,1,\dots,n-1$ of forward LIBORs, a collection of mutually equivalent probability measures $\mathbb{P}_{T_j},\ j=1,\dots,n,$ and a family $W^{T_j},\ j=1,2,\dots,n$ of processes in such a way that: (i) for any $j=1,2,\dots,n$ the process W^{T_j} follows a d-dimensional standard Brownian motion under the probability measure $\mathbb{P}_{T_j}$, (ii) for any $j=0,1,\dots,n-1$, the forward LIBOR $L(\cdot,T_j)$ satisfies the SDE

$$dL(t,T_j)=L(t,T_j)\,\lambda(t,T_j)\cdot dW_t^{T_{j+1}},\quad \forall\, t\in[0,T_j], \tag{19}$$

with the initial condition

$$L(0,T_j)=\frac{B(0,T_j)-B(0,T_{j+1})}{\delta_{j+1}B(0,T_{j+1})}.$$

As already mentioned, the construction of the model is based on backward induction, therefore we start by defining the forward LIBOR with the longest maturity, i.e., T_{n-1}. We postulate that $L(\cdot,T_{n-1})=L(\cdot,T_1^*)$ is governed under the underlying probability measure $\mathbb{P}$ by the following SDE[6]

$$dL(t,T_1^*)=L(t,T_1^*)\,\lambda(t,T_1^*)\cdot dW_t$$

with the initial condition

$$L(0,T_1^*)=\frac{B(0,T_1^*)-B(0,T^*)}{\delta_n B(0,T^*)}.$$

Put another way, we have

$$L(t,T_1^*)=\frac{B(0,T_1^*)-B(0,T^*)}{\delta_n B(0,T^*)}\,\mathcal{E}_t\Big(\int_0^{\cdot}\lambda(u,T_1^*)\cdot dW_u\Big).$$

Since $B(0,T_1^*)>B(0,T^*)$, it is clear that the $L(\cdot,T_1^*)$ follows a strictly positive martingale under $\mathbb{P}_{T^*}=\mathbb{P}$. The next step is to define the forward LIBOR for the date T_2^*. For this purpose, we need to introduce first the forward probability measure for the date T_1^*. By definition, it is a probability measure $\mathbb{Q}$, which is equivalent to $\mathbb{P}$, and such that processes

$$U_2(t,T_k^*)=\frac{B(t,T_k^*)}{B(t,T_1^*)}$$

are $\mathbb{Q}$-local martingales. It is important to observe that the process $U_2(\cdot,T_k^*)$ admits the following representation

$$U_2(t,T_k^*)=\frac{U_1(t,T_k^*)}{1+\delta_n L(t,T_1^*)}.$$

Let us formulate an auxiliary result, which is a straightforward consequence of Itô's rule.

Lemma 2.2 *Let G and H be real-valued adapted processes, such that*

$$dG_t=\alpha_t\cdot dW_t,\quad dH_t=\beta_t\cdot dW_t.$$

Assume, in addition, that $H_t>-1$ for every t and denote $Y_t=(1+H_t)^{-1}$. Then

$$d(Y_tG_t)=Y_t\big(\alpha_t-Y_tG_t\beta_t\big)\cdot\big(dW_t-Y_t\beta_t\,dt\big).$$

It follows immediately from Lemma 2.2 that

$$dU_2(t,T_k^*)=\eta_t^k\cdot\Big(dW_t-\frac{\delta_n L(t,T_1^*)}{1+\delta_n L(t,T_1^*)}\,\lambda(t,T_1^*)\,dt\Big)$$

[6] Notice that, for simplicity, we have chosen the underlying probability measure $\mathbb{P}$ to play the role of the forward LIBOR measure for the date T^*. This choice is not essential, however.

for a certain process η^k. Therefore it is enough to find a probability measure under which the process

$$W_t^{T_1^*} := W_t - \int_0^t \frac{\delta_n L(u,T_1^*)}{1+\delta_n L(u,T_1^*)}\,\lambda(u,T_1^*)\,du = W_t - \int_0^t \gamma(u,T_1^*)\,du,$$

$t \in [0,T_1^*]$, follows a standard Brownian motion (the definition of $\gamma(\cdot,T_1^*)$ is clear from the context). This can be easily achieved using Girsanov's theorem, as we may put

$$\frac{d\mathbb{P}_{T_1^*}}{d\mathbb{P}} = \mathcal{E}_{T_1^*}\Big(\int_0^{\cdot} \gamma(u,T_1^*)\cdot dW_u\Big), \quad \mathbb{P}\text{-a.s.}$$

We are in a position to specify the dynamics of the forward LIBOR for the date T_2^* under $\mathbb{P}_{T_1^*}$, namely we postulate that

$$dL(t,T_2^*) = L(t,T_2^*)\,\lambda(t,T_2^*)\cdot dW_t^{T_1^*}$$

with the initial condition

$$L(0,T_2^*) = \frac{B(0,T_2^*) - B(0,T_1^*)}{\delta_{n-1}B(0,T_1^*)}\,.$$

Let us now assume that we have found processes $L(\cdot,T_1^*),\dots,L(\cdot,T_m^*)$. This means, in particular, that the forward LIBOR measure $\mathbb{P}_{T_{m-1}^*}$ and the associated Brownian motion $W^{T_{m-1}^*}$ are already specified. Our aim is to determine the forward LIBOR measure $\mathbb{P}_{T_m^*}$. It is easy to check that

$$U_{m+1}(t,T_k^*) := \frac{B(t,T_k^*)}{B(t,T_m^*)} = \frac{U_m(t,T_k^*)}{1+\delta_{n-m}L(t,T_m^*)}.$$

Using Lemma 2.2, we obtain the following relationship

$$W_t^{T_m^*} = W_t^{T_{m-1}^*} - \int_0^t \frac{\delta_{n-m}L(u,T_m^*)}{1+\delta_{n-m}L(u,T_m^*)}\,\lambda(u,T_m^*)\,du$$

for $t\in[0,T_m^*]$. The forward LIBOR measure $\mathbb{P}_{T_m^*}$ can thus be easily found using Girsanov's theorem. Finally, we define the process $L(\cdot,T_{m+1}^*)$ as the solution to the SDE

$$dL(t,T_{m+1}^*) = L(t,T_{m+1}^*)\,\lambda(t,T_{m+1}^*)\cdot dW_t^{T_m^*}$$

with the initial condition

$$L(0,T_{m+1}^*) = \frac{B(0,T_{m+1}^*) - B(0,T_m^*)}{\delta_{n-m}B(0,T_m^*)}\,.$$

Remarks. (i) It is not difficult to check that equality (6) is satisfied within the present setup.
(ii) If the volatility coefficient $\lambda(\cdot,T_m) : [0,T_n] \to \mathbb{R}^d$ is a deterministic function, then for each date $t\in[0,T_m]$ the random variable $L(t,T_m)$ has a lognormal probability law under the forward probability measure $\mathbb{P}_{T_{m+1}}$.

2.2.4 SDE for LIBORs under the Forward Measure

We now consider a collection of reset/settlement dates $0 < T_1 < T_2 < \dots < T_{n+1}$. We assume that each LIBOR $L_i(t) = L(t,T_i)$, $i = 1,2,\dots,n$, solves under the physical probability $\mathbb{P}$ the following stochastic differential equation (SDE)

$$dL_i(t) = L_i(t)\big(\mu_i(t)\,dt + \sigma_i(L_i(t),t)\,dW_t^i\big). \tag{20}$$

In this SDE, μ_i is the drift coefficient and σ_i represents the volatility coefficient. The drift coefficient should only depend on the forward LIBORs L_j, $j=1,2,\dots,n$, existing at time t, and should be sufficiently regular in order to ensure the existence and uniqueness of a solution to the SDE (20). In general, we have

$$\mu_i(t) = \mu_i(L_1(t), L_2(t),\dots,L_n(t),t).$$

By assumption, the one-dimensional Brownian motions $W^1, W^2, \ldots, W^n$ in (20) are correlated, and their instantaneous correlations are given by

$$d\langle W^i, W^j\rangle_t = \rho_{i,j}(t)\, dt \tag{21}$$

for every $i, j = 1, 2, \ldots, n$.

The forward measure $\mathbb{P}_{T_{n+1}}$ for maturity T_{n+1} uses the risk-free zero-coupon bond maturing at T_{n+1} as a numeraire. Using this new numeraire, the relative bond prices are

$$D_i(t) = B(t, T_i)/B(t, T_{n+1}),$$

which can also be expressed in terms of forward LIBORs, namely,

$$D_i(t) = \prod_{j=i}^{n} (1 + \delta_j L_j(t)).$$

Proposition 2.2 *The drift term $\widehat{\mu}_i(t)$ in the dynamics of $L_i(t) = L(t, T_i)$ under the forward measure $\mathbb{P}_{T_{n+1}}$ equals*

$$\widehat{\mu}_i(t) = -\sum_{j=i+1}^{n} \frac{\delta_{i+1} L_j(t)}{1 + \delta_j L_j(t)} \sigma_i(L_i(t), t)\sigma_j(L_j(t), t)\rho_{i,j}(t). \tag{22}$$

The stochastic differential equation for the forward LIBORs $L_1, L_2, \ldots, L_n$ under the forward measure $\mathbb{P}_{T_{n+1}}$ has the form

$$dL_i(t) = L_i(t) \left(-\sum_{j=i+1}^{n} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)} \sigma_i(L_i(t), t)\sigma_j(L_j(t), t)\rho_{i,j}(t)\, dt + \sigma_i(L_i(t), t)\, d\widehat{W}^i_t \right)$$

where $\widehat{W}^1, \widehat{W}^2, \ldots, \widehat{W}^n$ are Brownian motions under $\mathbb{P}_{T_{n+1}}$ with the instantaneous correlations given by

$$d\langle \widehat{W}^i, \widehat{W}^j\rangle_t = \rho_{i,j}(t)\, dt \tag{23}$$

$i, j = 1, 2, \ldots, n$.

Proof. Using Girsanov's theorem, we get from (20), for $i = 1, 2, \ldots, n$,

$$dL_i(t) = L_i(t)\big(\widehat{\mu}_i(t)\, dt + \sigma_i(L_i(t), t)\, d\widehat{W}^i_t\big), \tag{24}$$

where $\widehat{W}^1, \widehat{W}^2, \ldots, \widehat{W}^n$ are Brownian motions under $\mathbb{P}_{T_{n+1}}$ with the instantaneous correlations given by (23) (as is well known, an equivalent change of a probability measure preserves the correlations between Brownian motions). The drift coefficients $\widehat{\mu}_i(t)$ are not yet specified, however.

The derivation of drift coefficients $\widehat{\mu}_i(t)$ is based on the requirement that the relative bond prices D_i have the martingale property under the forward measure $\mathbb{P}_{T_{n+1}}$. Applying the Ito's formula to the equality

$$D_i(t) = D_{i+1}(t)\,(1 + \delta_i L_i(t)), \tag{25}$$

we obtain

$$dD_i(t) = (1 + \delta_i L_i(t))dD_{i+1}(t) + \delta_i D_{i+1}(t)dL_i(t) + \delta_i d\langle D_{i+1}, L_i\rangle_t.$$

Since the relative prices D_i and D_{i+1} follow martingales under $\mathbb{P}_{T_{n+1}}$, and the finite variation terms in the differential $dD_i(t)$ should vanish, we find that the drift $\widehat{\mu}_i(t)$ should satisfy

$$D_{i+1}(t)\widehat{\mu}_i(t)L_i(t)\, dt = -d\langle D_{i+1}, L_i\rangle_t. \tag{26}$$

To establish (22), it suffices to compute the cross-variation $\langle D_{i+1}, L_i \rangle$. To this end, we shall find the martingale component in the canonical decomposition of D_{i+1}. Since

$$D_{i+1}(t) = \prod_{j=i+1}^{n} \left(1 + \delta_j L_j(t)\right),$$

we have

$$\begin{aligned} dD_{i+1}(t) &= \sum_{j=i+1}^{n} \prod_{k=i+1,\, k \neq j}^{n} \left(1 + \delta_k L_k(t)\right) d\left(1 + \delta_j L_j(t)\right) + A_t \\ &= \sum_{j=i+1}^{n} \prod_{k=i+1,\, k \neq j}^{n} \left(1 + \delta_k L_k(t)\right) \delta_j \, dL_j(t) + A_t \\ &= D_{i+1}(t) \sum_{j=i+1}^{n} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)} \, \sigma_j(L_j(t), t) \, d\widehat{W}_t^j + B_t, \end{aligned}$$

where A and B are some continuous processes of finite variation. Consequently,

$$d\langle D_{i+1}, L_i \rangle_t = D_{i+1}(t) L_i(t) \sum_{j=i+1}^{n} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)} \, \sigma_i(L_i(t), t) \sigma_j(L_j(t), t) \rho_{i,j}(t) \, dt$$

Inserting the last equality in (26), we conclude that

$$\widehat{\mu}_i(t) = - \sum_{j=i+1}^{n} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)} \sigma_i(L_i(t), t) \sigma_j(L_j(t), t) \rho_{i,j}(t), \tag{27}$$

as expected. □

2.2.5 Jamshidian's Approach

The backward induction approach to modelling of forward LIBORs presented in the preceding section was re-examined and essentially generalized by Jamshidian (1997). In this section, we present briefly his approach to the modelling of forward LIBORs. As made apparent in the preceding section, in the direct modelling of LIBORs, no explicit reference is made to the bond price processes, which are used to formally define a forward LIBOR through equality (12). Nevertheless, to explain the idea that underpins Jamshidian's approach, we shall temporarily assume that we are given a family of bond prices $B(t, T_j)$ for the future dates T_j, $j = 1, 2, \ldots, n$. By definition, the *spot LIBOR measure* is that probability measure equivalent to $\mathbb{P}$, under which all relative bond prices are local martingales, when the price process obtained by rolling over single-period bonds, is taken as a numeraire. The existence of such a measure can be either postulated, or derived from other conditions.[7] Let us put, for $t \in [0, T^*]$ (as before $T_{-1} = 0$)

$$G_t = B(t, T_{m(t)}) \prod_{j=0}^{m(t)} B^{-1}(T_{j-1}, T_j), \tag{28}$$

where

$$m(t) = \inf\{k = 0, 1, \ldots \mid \sum_{i=0}^{k} \delta_i \geq t\} = \inf\{k = 0, 1, \ldots \mid T_k \geq t\}.$$

It is easily seen that G_t represents the wealth at time t of a portfolio which starts at time 0 with one unit of cash invested in a zero-coupon bond of maturity T_0, and whose wealth is then reinvested at each date T_j, $j = 0, 1, \ldots, n-1$, in zero-coupon bonds which mature at the next date; that is, T_{j+1}.

[7] One may assume, e.g., that bond prices $B(t, T_j)$ satisfy the *weak no-arbitrage condition,* meaning that there exists a probability measure $\widetilde{\mathbb{P}}$, equivalent to $\mathbb{P}$, and such that all processes $B(t, T_k)/B(t, T^*)$ are $\widetilde{\mathbb{P}}$-local martingales.

Definition 2.3 A *spot LIBOR measure* $\mathbb{P}^L$ is a probability measure on $(\Omega, \mathcal{F}_{T^*})$ which is equivalent to $\mathbb{P}$, and such that for any $j = 0, 1, \ldots, n$ the relative bond price $B(t, T_j)/G_t$ follows a local martingale under $\mathbb{P}^L$.

Note that

$$B(t, T_{k+1})/G_t = \prod_{j=0}^{m(t)} \big(1 + \delta_j L(T_{j-1}, T_{j-1})\big)^{-1} \prod_{j=m(t)+1}^{k} \big(1 + \delta_j L(t, T_{j-1})\big)$$

so that all relative bond prices $B(t, T_j)/G_t$, $j = 0, 1, \ldots, n$ are uniquely determined by a collection of forward LIBORs. In this sense, G is the correct choice of the reference price process in the present setting.

2.2.6 SDE for LIBORs under the Spot LIBOR Measure

We shall now concentrate on the derivation of the dynamics under the probability measure $\mathbb{P}^L$ of forward LIBORs $L(\cdot, T_j)$, $j = 0, 1, \ldots, n-1$. Our aim is to show that these dynamics involve only the volatilities of forward LIBOR rates (as opposed to volatilities of bond prices or other processes). Therefore, it is possible to define the whole family of forward LIBORs simultaneously under one probability measure (of course, this feature can also be deduced from the preceding construction).

Proposition 2.3 *Processes $L(\cdot, T_j)$, $j = 0, 1, \ldots, n-1$, satisfy the SDE*

$$dL(t, T_j) = \sum_{k=m(t)}^{j} \frac{\delta_{k+1}\zeta(t, T_k) \cdot \zeta(t, T_j)}{1 + \delta_{k+1} L(t, T_k)}\, dt + \zeta(t, T_j) \cdot dW_t^L,$$

where the process W^L is a d-dimensional standard Brownian motion under the spot LIBOR measure $\mathbb{P}^L$ and $\zeta(t, T_j)$ is some $\mathbb{F}$-adapted process.

Proof. To facilitate the derivation of the dynamics of $L(\cdot, T_j)$, we postulate temporarily that bond prices $B(t, T_j)$ follow Itô processes under the underlying probability measure $\mathbb{P}$, more explicitly

$$dB(t, T_j) = B(t, T_j)\big(a(t, T_j)\, dt + b(t, T_j) \cdot dW_t\big) \tag{29}$$

for every $j = 0, 1, \ldots, n$, where W is a d-dimensional standard Brownian motion under $\mathbb{P}$. It should be stressed, however, that we do not assume here that $\mathbb{P}$ is a forward (or spot) martingale measure. Combining (28) with (29), we obtain

$$dG_t = G_t\big(a(t, T_{m(t)})\, dt + b(t, T_{m(t)}) \cdot dW_t\big). \tag{30}$$

Furthermore, by applying Itô's rule to equality

$$1 + \delta_{j+1} L(t, T_j) = \frac{B(t, T_j)}{B(t, T_{j+1})}, \tag{31}$$

we find that

$$dL(t, T_j) = \mu(t, T_j)\, dt + \zeta(t, T_j) \cdot dW_t,$$

where

$$\mu(t, T_j) = \frac{B(t, T_j)}{\delta_{j+1} B(t, T_{j+1})}\big(a(t, T_j) - a(t, T_{j+1})\big) - \zeta(t, T_j) b(t, T_{j+1})$$

and

$$\zeta(t, T_j) = \frac{B(t, T_j)}{\delta_{j+1} B(t, T_{j+1})}\big(b(t, T_j) - b(t, T_{j+1})\big). \tag{32}$$

Using (31) and the last formula, we arrive at the following relationship

$$b(t, T_{m(t)}) - b(t, T_{j+1}) = \sum_{k=m(t)}^{j} \frac{\delta_{k+1}\zeta(t, T_k)}{1 + \delta_{k+1}L(t, T_k)}. \tag{33}$$

By definition of a spot LIBOR measure $\mathbb{P}^L$, each relative price $B(t, T_j)/G_t$ follows a local martingale under $\mathbb{P}^L$. Since, in addition, $\mathbb{P}^L$ is assumed to be equivalent to $\mathbb{P}$, it is clear that it is given by the Doléans exponential, that is

$$\frac{d\mathbb{P}^L}{d\mathbb{P}} = \mathcal{E}_{T^*}\Big(\int_0^{\cdot} h_u \cdot dW_u\Big), \quad \mathbb{P}\text{-a.s.}$$

for some adapted process h. It it not hard to check, using Itô's rule, that h necessarily satisfies, for $t \in [0, T_j]$,

$$a(t, T_j) - a(t, T_{m(t)}) = \big(b(t, T_{m(t)}) - h_t\big) \cdot \big(b(t, T_j) - b(t, T_{m(t)})\big)$$

for every $j = 0, 1, \ldots, n$. Combining (32) with the last formula, we obtain

$$\frac{B(t, T_j)}{\delta_{j+1}B(t, T_{j+1})}\big(a(t, T_j) - a(t, T_{j+1})\big) = \zeta(t, T_j) \cdot \big(b(t, T_{m(t)}) - h_t\big),$$

and this in turn yields

$$dL(t, T_j) = \zeta(t, T_j) \cdot \Big(\big(b(t, T_{m(t)}) - b(t, T_{j+1}) - h_t\big)\, dt + dW_t\Big).$$

Using (33), we conclude that process $L(\cdot, T_j)$ satisfies

$$dL(t, T_j) = \sum_{k=m(t)}^{j} \frac{\delta_{k+1}\zeta(t, T_k) \cdot \zeta(t, T_j)}{1 + \delta_{k+1}L(t, T_k)}\, dt + \zeta(t, T_j) \cdot dW_t^L,$$

where the process $W_t^L = W_t - \int_0^t h_u\, du$ follows a d-dimensional standard Brownian motion under the spot LIBOR measure $\mathbb{P}^L$. □

To further specify the model, we assume that processes $\zeta(t, T_j)$, $j = 0, 1, \ldots, n-1$, have the following form, for $t \in [0, T_j]$,

$$\zeta(t, T_j) = \lambda_j\big(t, L(t, T_j), L(t, T_{j+1}), \ldots, L(t, T_{n-1})\big),$$

where $\lambda_j : [0, T_j] \times \mathbb{R}^{n-j+1} \to \mathbb{R}^d$ are given functions. In this way, we obtain a system of SDEs

$$dL(t, T_j) = \sum_{k=m(t)}^{j} \frac{\delta_{k+1}\lambda_k(t, L_k(t)) \cdot \lambda_j(t, L_j(t))}{1 + \delta_{k+1}L(t, T_k)}\, dt + \lambda_j(t, L_j(t)) \cdot dW_t^L,$$

where we write $L_j(t) = (L(t, T_j), L(t, T_{j+1}), \ldots, L(t, T_{n-1}))$. Under mild regularity assumptions, this system can be solved recursively, starting from $L(\cdot, T_{n-1})$. The lognormal model of forward LIBORs (LLM, for short) corresponds to the choice of $\zeta(t, T_j) = \lambda(t, T_j)L(t, T_j)$, where $\lambda(\cdot, T_j) : [0, T_j] \to \mathbb{R}^d$ is a deterministic function for every j.

2.2.7 Alternative Derivation of the SDE for LIBORs

We present below an alternative derivation of dynamics of forward LIBORs under the spot LIBOR measure. We now adopt the convention that the reset/settlement dates are $0 < T_1 < T_2 < \cdots < T_{n+1}$, so that

$$G_t = B(t, T_{m(t)}) \prod_{j=1}^{m(t)} B^{-1}(T_{j-1}, T_j), \tag{34}$$

where $T_0 = 0$ and

$$m(t) = \inf\{k = 1, 2, \dots \mid T_k \geq t\}.$$

It is useful to observe that the relative bond prices satisfy

$$\widetilde{D}(t, T_i) = B(t, T_i)/G_t = \prod_{j=0}^{m(t)-1} \frac{1}{1 + \delta_j L_j(T_j)} \prod_{j=m(t)}^{i-1} \frac{1}{1 + \delta_j L_j(t)}. \tag{35}$$

Proposition 2.4 *The stochastic differential equation for the forward LIBORs $L_1, L_2, \dots, L_n$ under the spot LIBOR measure $\mathbb{P}^L$ is given by*

$$dL_i(t) = L_i(t) \left(\sum_{j=m(t)}^{i} \frac{\delta_j L_j(t) \sigma_j(L_j(t), t) \sigma_i(L_i(t), t) \rho_{i,j}(t)}{1 + \delta_j L_j(t)}\, dt + \sigma_i(L_i(t), t)\, d\widetilde{W}_t^i \right) \tag{36}$$

where $\widetilde{W}^1, \widetilde{W}^2, \dots, \widetilde{W}^n$ are one-dimensional correlated Brownian motions under $\mathbb{P}^L$ with the instantaneous correlations given by

$$d\langle \widetilde{W}^i, \widetilde{W}^j \rangle_t = d\langle W^i, W^j \rangle_t = \rho_{i,j}(t)\, dt \tag{37}$$

for every $1 \leq i, j \leq n$.

Proof. Since the relative bond price $\widetilde{D}_i(t) = \widetilde{D}(t, T_i)$ should follow a martingale under the spot LIBOR measure $\mathbb{P}^L$, we require that the drift term in the dynamics of $\widetilde{D}_i$ under $\mathbb{P}^L$ equals zero. The proof is based on an application of Girsanov's theorem. We do not present the detailed proof here, and we shall focus on the derivation of the drift coefficient $\widetilde{\mu}_i$ in the dynamics

$$dL_i(t) = \widetilde{\mu}_i(t) L_i(t)\, dt + \sigma_i(L_i, t) L_i(t)\, d\widetilde{W}_t^i \tag{38}$$

of the forward LIBOR L_i under $\mathbb{P}^L$. In formula (38), the process $\widetilde{W}^i$ is a Brownian motion under the spot LIBOR measure $\mathbb{P}^L$.

We start by combining (35) with (20). An application of Itô's Lemma yields

$$\begin{aligned}
d\widetilde{D}_i(t) &= d\left(\prod_{j=0}^{m(t)-1} \frac{1}{1 + \delta_j L_j(T_j)} \prod_{j=m(t)}^{i-1} \frac{1}{1 + \delta_j L_j(t)} \right) \\
&= -\widetilde{D}_i(t) \sum_{j=m(t)}^{i-1} \left(\frac{\delta_j}{1 + \delta_j L_j(t)}\, dL_j(t) - \frac{\delta_j^2}{(1 + \delta_j L_j(t))^2}\, d\langle L_j \rangle_t \right) \\
&= -\widetilde{D}_i(t) \sum_{j=m(t)}^{i-1} \frac{\delta_j}{1 + \delta_j L_j(t)} \Big(\mu_j(t) L_j(t)\, dt + \sigma_j(L_j(t), t) L_j(t)\, dW_t^j \Big) \\
&\quad -\widetilde{D}_i(t) \sum_{j=m(t)}^{i-1} \frac{\delta_j^2}{(1 + \delta_j L_j(t))^2}\, d\langle L_j \rangle_t \\
&= -\widetilde{D}_i(t) \sum_{j=m(t)}^{i-1} \Big(\frac{\delta_j}{1 + \delta_j L_j(t)}\, \mu_j(t) L_j(t) - \frac{\delta_j^2}{(1 + \delta_j L_j(t))^2}\, \sigma_j^2(L_j(t), t) L_j^2(t) \Big) dt \\
&\quad -\widetilde{D}_i(t) \sum_{j=m(t)}^{i-1} \frac{\delta_j}{1 + \delta_j L_j(t)}\, \sigma_j(L_j(t), t) L_j(t)\, dW_t^j.
\end{aligned}$$

From the last formula, we deduce that the local martingale part of $\widetilde{D}_i$ is given by (in a differential form)

$$-\widetilde{D}_i(t) \sum_{j=m(t)}^{i-1} \frac{\delta_j}{1 + \delta_j L_j(t)}\, \sigma_j(L_j(t), t) L_j(t)\, dW_t^j.$$

Assuming that we can apply Girsanov's theorem, we see that the assumption that $\widetilde{D}_i$ is a martingale under $\mathbb{P}^L$ leads to the following stochastic differential equation

$$d\widetilde{D}_i(t) = -\widetilde{D}_i(t) \sum_{j=m(t)}^{i-1} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\, \sigma_j(L_j(t),t)\, d\widetilde{W}_t^j, \tag{39}$$

where $\widetilde{W}^1, \widetilde{W}^2, \ldots, \widetilde{W}^n$ are one-dimensional Brownian motions under $\mathbb{P}^L$ with the instantaneous correlations given by formula (37).

To derive the drift term in dynamics of L_i under the probability $\mathbb{P}^L$, we shall use the relationship $\widetilde{D}_i(t) = \widetilde{D}_{i+1}(t)(1+\delta_i L_i(t))$. From this identity, we find that $L_i(t)\widetilde{D}_{i+1}(t)$ is a martingale under $\mathbb{P}^L$ if and only if $\widetilde{D}_i(t)$ is a martingale under this probability. Therefore, the bounded variation part of the differential $d(L_i(t)\widetilde{D}_{i+1}(t))$ should vanish.

Using Itô's lemma and equation (39), we obtain

$$\begin{aligned} d(L_i(t)\widetilde{D}_{i+1}(t)) &= \widetilde{D}_{i+1}(t)\, dL_i(t) + L_i(t)\, d\widetilde{D}_{i+1}(t) \\ &\quad - L_i(t)\widetilde{D}_{i+1}(t) \sum_{j=m(t)}^{i} \frac{\delta_j}{1+\delta_j L_j(t)}\, \sigma_j(L_j(t),t)\sigma_i(L_i(t),t)\rho_{i,j}(t)\, dt \\ &= L_i(t)\widetilde{D}_{i+1}(t) \left(\widetilde{\mu}_i(t) - \sum_{j=m(t)}^{i} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\, \sigma_j(L_j(t),t)\sigma_i(L_i(t),t)\rho_{i,j}(t) \right) dt \\ &\quad + L_i(t)\widetilde{D}_{i+1}(t)\sigma_i(L_i(t),t)\, d\widetilde{W}_t^i + L_i(t)\, d\widetilde{D}_{i+1}(t). \end{aligned}$$

The process $L_i(t)\widetilde{D}_{i+1}(t)$ is a (local) martingale if and only if $\widetilde{D}_i(t)$ is a (local) martingale, and the continuous process of bounded variation in the previous equation is identical to zero, i.e.,

$$\widetilde{\mu}_i(t) = \sum_{j=m(t)}^{i} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\, \sigma_j(L_j(t),t)\sigma_i(L_i(t),t)\rho_{i,j}(t).$$

This completes the proof of the proposition. □

2.3 Caps and Floors

An *interest rate cap* (known also as a *ceiling rate agreement*) is a contractual arrangement where the grantor (seller) has an obligation to pay cash to the holder (buyer) if a particular interest rate exceeds a mutually agreed level at some future date or dates. Similarly, in an *interest rate floor,* the grantor has an obligation to pay cash to the holder if the interest rate is below a preassigned level. When cash is paid to the holder, the holder's net position is equivalent to borrowing (or depositing) at a rate fixed at that agreed level. This assumes that the holder of a cap (or floor) agreement also holds an underlying asset (such as a deposit) or an underlying liability (such as a loan). Finally, the holder is not affected by the agreement if the interest rate is ultimately more favorable to him than the agreed level. This feature of a cap (or floor) agreement makes it similar to an option. Specifically, a *forward start cap* (or a *forward start floor*) is a strip of caplets (floorlets), each of which is a call (put) option on a forward rate, respectively. Let us denote by κ and by δ_j the cap strike rate and the length of the accrual period, respectively. We shall check that an interest rate caplet (i.e., one leg of a cap) may also be seen as a put option with strike price 1 (per dollar of notional principal) which expires at the caplet start day on a discount bond with face value $1+\kappa\delta_j$ which matures at the caplet end date.

Similarly to swap agreements, interest rate caps and floors may be settled either *in arrears* or *in advance.* In a forward cap or floor, which starts at time T_0, and is settled in arrears at dates T_j, $j = 1, 2, \ldots, n$, the cash flows at times T_j are $N_p(L(T_{j-1}) - \kappa)^+\delta_j$ and $N_p(\kappa - L(T_{j-1}))^+\delta_j$, respectively, where N_p stands for the notional principal (recall that $\delta_j = T_j - T_{j-1}$). As usual, the rate $L(T_{j-1}) = L(T_{j-1}, T_{j-1})$ is determined at the reset date T_{j-1}, and it satisfies

$$B(T_{j-1}, T_j)^{-1} = 1 + \delta_j L(T_{j-1}). \tag{40}$$

The price at time $t \le T_0$ of a *forward cap,* denoted by $\mathbf{FC}_t$, is (we set $N_p = 1$)

$$\begin{aligned} \mathbf{FC}_t &= \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*} \Big(\frac{B_t}{B_{T_j}} (L(T_{j-1}) - \kappa)^+ \delta_j \,\Big|\, \mathcal{F}_t \Big) \\ &= \sum_{j=1}^{n} B(t, T_j)\, \mathbb{E}_{\mathbb{P}_{T_j}} \Big((L(T_{j-1}) - \kappa)^+ \delta_j \,\Big|\, \mathcal{F}_t \Big). \end{aligned} \tag{41}$$

On the other hand, since the cash flow of the j^{th} caplet at time T_j is manifestly a $\mathcal{F}_{T_{j-1}}$-measurable random variable, we may directly express the value of the cap in terms of expectations under forward measures $\mathbb{P}_{T_{j-1}}$, $j = 1, 2, \dots, n$. Indeed, we have

$$\mathbf{FC}_t = \sum_{j=1}^{n} B(t, T_{j-1})\, \mathbb{E}_{\mathbb{P}_{T_{j-1}}} \Big(B(T_{j-1}, T_j)(L(T_{j-1}) - \kappa)^+ \delta_j \,\Big|\, \mathcal{F}_t \Big). \tag{42}$$

Consequently, using (40) we get equality

$$\mathbf{FC}_t = \sum_{j=1}^{n} B(t, T_{j-1})\, \mathbb{E}_{\mathbb{P}_{T_{j-1}}} \Big(\big(1 - \widetilde{\delta}_j B(T_{j-1}, T_j)\big)^+ \,\Big|\, \mathcal{F}_t \Big), \tag{43}$$

which is valid for every $t \in [0, T]$. It is apparent that a caplet is essentially equivalent to a put option on a zero-coupon bond; it may also be seen as an option on a single-period swap.

The equivalence of a cap and a put option on a zero-coupon bond can be explained in an intuitive way. For this purpose, it is enough to examine two basic features of both contracts: the exercise set and the payoff value. Let us consider the j^{th} caplet. A caplet is exercised at time T_{j-1} if and only if $L(T_{j-1}) - \kappa > 0$, or equivalently, if

$$B(T_{j-1}, T_j)^{-1} = 1 + L(T_{j-1})(T_j - T_{j-1}) > 1 + \kappa\delta_j = \widetilde{\delta}_j.$$

The last inequality holds whenever $\widetilde{\delta}_j B(T_{j-1}, T_j) < 1$. This shows that both of the considered options are exercised in the same circumstances. If exercised, the caplet pays $\delta_j(L(T_{j-1}) - \kappa)$ at time T_j, or equivalently

$$\delta_j B(T_{j-1}, T_j)(L(T_{j-1}) - \kappa) = 1 - \widetilde{\delta}_j B(T_{j-1}, T_j) = \widetilde{\delta}_j \left(\widetilde{\delta}_j^{-1} - B(T_{j-1}, T_j) \right)$$

at time T_{j-1}. This shows once again that the j^{th} caplet, with strike level κ and nominal value 1, is essentially equivalent to a put option with strike price $(1 + \kappa\delta_j)^{-1}$ and nominal value $\widetilde{\delta}_j = (1 + \kappa\delta_j)$ written on the corresponding zero-coupon bond with maturity T_j.

The analysis of a floor contract can be done long the similar lines. By definition, the j^{th} floorlet pays $(\kappa - L(T_{j-1}))^+$ at time T_j. Therefore,

$$\mathbf{FF}_t = \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*} \Big(\frac{B_t}{B_{T_j}} (\kappa - L(T_{j-1}))^+ \delta_j \,\Big|\, \mathcal{F}_t \Big), \tag{44}$$

but also

$$\mathbf{FF}_t = \sum_{j=1}^{n} B(t, T_{j-1})\, \mathbb{E}_{\mathbb{P}_{T_{j-1}}} \Big(\big(\widetilde{\delta}_j B(T_{j-1}, T_j) - 1\big)^+ \,\Big|\, \mathcal{F}_t \Big). \tag{45}$$

Combining (41) with (44) (or (43) with (45)), we obtain the following cap-floor parity relationship

$$\mathbf{FC}_t - \mathbf{FF}_t = \sum_{j=1}^{n} \big(B(t, T_{j-1}) - \widetilde{\delta}_j B(t, T_j) \big). \tag{46}$$

This relationship is also an immediate consequence of the no-arbitrage property, so that it does not depend on a model choice.

2.3.1 Market Formula for Caps and Floors

The main motivation for the introduction of a lognormal model of LIBORs is the market practice of pricing caps and swaptions by means of Black-Scholes-like formulae. For this reason, we shall first describe how market practitioners value caps. The formulae commonly used by practitioners assume that the underlying instrument follows a geometric Brownian motion under some probability measure, $\mathbb{Q}$ say. Since the formal definition of this probability measure is not available, we shall informally refer to $\mathbb{Q}$ as the *market probability*.

Let us consider an interest rate cap with expiry date T and fixed strike level κ. Market practice is to price the option assuming that the underlying forward interest rate process is lognormally distributed with zero drift. Let us first consider a caplet – that is, one leg of a cap. Assume that the forward LIBOR $L(t,T)$, $t \in [0,T]$, for the accrual period of length δ follows a geometric Brownian motion under the "market probability", $\mathbb{Q}$ say. More specifically

$$dL(t,T) = L(t,T)\sigma \, dW_t, \tag{47}$$

where W follows a one-dimensional standard Brownian motion under $\mathbb{Q}$, and σ is a strictly positive constant. The unique solution of (47) is

$$L(t,T) = L(0,T) \exp\big(\sigma W_t - \tfrac{1}{2}\sigma^2 t^2\big), \quad \forall\, t \in [0,T], \tag{48}$$

where the initial condition is derived from the yield curve $Y(0,T)$, namely

$$1 + \delta L(0,T) = \frac{B(0,T)}{B(0,T+\delta)} = \exp\Big((T+\delta) Y(0,T+\delta) - TY(0,T) \Big).$$

The "market price" at time t of a caplet with expiry date T and strike level κ is calculated by means of the formula

$$\mathbf{FC}_t = \delta B(t, T+\delta) \, \mathbb{E}_{\mathbb{Q}} \Big((L(T,T) - \kappa)^+ \,\Big|\, \mathcal{F}_t \Big).$$

More explicitly, for any $t \in [0,T]$ we have

$$\mathbf{FC}_t = \delta B(t, T+\delta) \Big(L(t,T) N\big(\widehat{e}_1(t,T)\big) - \kappa N\big(\widehat{e}_2(t,T)\big) \Big), \tag{49}$$

where N is the standard Gaussian cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \, dz, \quad \forall\, x \in \mathbb{R},$$

and

$$\widehat{e}_{1,2}(t,T) = \frac{\ln(L(t,T)/\kappa) \pm \frac{1}{2} \widehat{v}_0^2(t,T)}{\widehat{v}_0(t,T)}$$

with $\widehat{v}_0^2(t,T) = \sigma^2 (T-t)$. This means that market practitioners price caplets using Black's formula, with discount from the settlement date $T + \delta$.

A cap settled in arrears at times T_j, $j = 1, 2, \ldots, n$, where $T_j - T_{j-1} = \delta_j$, $T_0 = T$, is priced by the formula

$$\mathbf{FC}_t = \sum_{j=1}^{n} \delta_j B(t, T_j) \Big(L(t, T_{j-1}) N\big(\widehat{e}_1^{\,j}(t)\big) - \kappa N\big(\widehat{e}_2^{\,j}(t)\big) \Big), \tag{50}$$

where for every $j = 0, 1, \ldots, n-1$

$$\widehat{e}^{\,j}_{1,2}(t) = \frac{\ln(L(t,T_{j-1})/\kappa) \pm \frac{1}{2}\,\widehat{v}^{\,2}_j(t)}{\widehat{v}_j(t)} \tag{51}$$

and $\widehat{v}^{\,2}_j(t) = (T_{j-1} - t)\sigma_j^2$ for some constants σ_j, $j = 1, 2, \ldots, n$. Apparently, the market assumes that for any maturity T_j, the corresponding forward LIBOR has a lognormal probability law under the "market probability". The value of a floor can be easily derived by combining (50)–(51) with the cap-floor parity relationship (46). As we shall see in what follows, the valuation formulae obtained for caps and floors in the lognormal model of forward LIBORs agree with the market practice.

2.3.2 Valuation in the LLM

We shall now examine the valuation of caps within the lognormal LIBOR model of Section 2.2.3. The dynamics of the forward LIBOR rate $L(t, T_{j-1})$ under the forward probability measure $\mathbb{P}_{T_j}$ are

$$dL(t, T_{j-1}) = L(t, T_{j-1})\,\lambda(t, T_{j-1}) \cdot dW^{T_j}_t, \tag{52}$$

where W^{T_j} follows a d-dimensional Brownian motion under the forward measure $\mathbb{P}_{T_j}$, and $\lambda(\cdot, T_{j-1}) : [0, T_{j-1}] \to \mathbb{R}^d$ is a deterministic function. Consequently, for every $t \in [0, T_{j-1}]$ we have

$$L(t, T_{j-1}) = L(0, T_{j-1})\mathcal{E}_t\Big(\int_0^{\cdot} \lambda(u, T_{j-1}) \cdot dW^{T_j}_u\Big).$$

In the present setup, the cap valuation formula (53) was first established by Miltersen et al. (1997), who focused on the dynamics of the forward LIBOR for a given date. Equality (53) was subsequently rederived through a probabilistic approach in Goldys (1997) and Rady (1997). Finally, the same result was established by means of the forward measure approach in Brace et al. (1997). The following proposition is a consequence of formula (42), combined with the dynamics (52). Let N stand for the standard Gaussian cumulative distribution function.

Proposition 2.5 *Consider a cap with strike level κ, settled in arrears at times T_j, $j = 1, 2, \ldots, n$. Assuming the lognormal LIBOR model, the price of a cap at time $t \in [0, T]$ equals*

$$\mathbf{FC}_t = \sum_{j=1}^{n} \delta_j B(t, T_j)\Big(L(t, T_{j-1})N\big(\widetilde{e}^{\,j}_1(t)\big) - \kappa N\big(\widetilde{e}^{\,j}_2(t)\big)\Big) = \sum_{j=1}^{n} \mathbf{FC}^{\,j}_t, \tag{53}$$

where $\mathbf{FC}^{\,j}_t$ stands for the price at time t of the j^{th} caplet for $j = 1, 2, \ldots, n$, and

$$\widetilde{e}^{\,j}_{1,2}(t) = \frac{\ln(L(t,T_{j-1})/\kappa) \pm \frac{1}{2}\,\widetilde{v}^{\,2}_j(t)}{\widetilde{v}_j(t)}$$

with

$$\widetilde{v}^{\,2}_j(t) = \int_t^{T_{j-1}} |\lambda(u, T_{j-1})|^2\,du.$$

Proof. We fix j and we consider the j^{th} caplet. It is clear that its payoff at time T_j admits the representation

$$\mathbf{FC}^{\,j}_{T_j} = \delta_j (L(T_{j-1}) - \kappa)^+ = \delta_j L(T_{j-1})\,\mathbb{1}_D - \delta_j \kappa\,\mathbb{1}_D, \tag{54}$$

where $D = \{L(T_{j-1}) > \kappa\}$ is the exercise set. Since the caplet settles at time T_j, it is convenient to use the forward measure $\mathbb{P}_{T_j}$ to find its arbitrage price. We have

$$\mathbf{FC}^{\,j}_t = B(t, T_j)\mathbb{E}_{\mathbb{P}_{T_j}}\big(\mathbf{FC}^{\,j}_{T_j}\,|\,\mathcal{F}_t\big), \quad \forall\, t \in [0, T_j].$$

Obviously, it is enough to find the value of a caplet for $t \in [0, T_{j-1}]$. In view of (54), it is clear that we need to evaluate the following conditional expectations

$$\begin{aligned}\mathbf{FC}_t^j &= \delta_j B(t,T_j)\,\mathbb{E}_{\mathbb{P}_{T_j}}\big(L(T_{j-1})\,\mathbb{1}_D \,\big|\, \mathcal{F}_t\big) - \kappa\delta_j B(t,T_j)\,\mathbb{P}_{T_j}(D\,|\,\mathcal{F}_t)\\ &= \delta_j B(t,T_j)(I_1 - I_2),\end{aligned}$$

where the meaning of I_1 and I_2 is clear from the context. Recall that $L(T_{j-1})$ is given by the formula

$$L(T_{j-1}) = L(t,T_{j-1})\exp\Big(\int_t^{T_{j-1}} \lambda(u,T_{j-1})\cdot dW_u^{T_j} - \frac{1}{2}\int_t^{T_{j-1}} |\lambda(u,T_{j-1})|^2\,du\Big).$$

Since $\lambda(\cdot, T_{j-1})$ is a deterministic function, the probability law under $\mathbb{P}_{T_j}$ of the Itô integral

$$\zeta(t,T_{j-1}) = \int_t^{T_{j-1}} \lambda(u,T_{j-1})\cdot dW_u^{T_j}$$

is Gaussian, with zero mean and the variance

$$\mathrm{Var}_{\mathbb{P}_{T_j}}\,(\zeta(t,T_{j-1})) = \int_t^{T_{j-1}} |\lambda(u,T_{j-1})|^2\,du.$$

Therefore, it is straightforward to show that[8]

$$I_2 = \kappa\, N\left(\frac{\ln(L(t,T_{j-1}) - \ln\kappa - \frac{1}{2}v_j^2(t)}{v_j(t)}\right).$$

To evaluate I_1, we introduce an auxiliary probability measure $\widehat{\mathbb{P}}_{T_j}$, equivalent to $\mathbb{P}_{T_j}$ on $(\Omega, \mathcal{F}_{T_{j-1}})$, by setting

$$\frac{d\widehat{\mathbb{P}}_{T_j}}{d\mathbb{P}_{T_j}} = \mathcal{E}_{T_{j-1}}\Big(\int_0^{\cdot} \lambda(u,T_{j-1})\cdot dW_u^{T_j}\Big).$$

Then the process $\widehat{W}^{T_j}$ given by the formula

$$\widehat{W}_t^{T_j} = W_t^{T_j} - \int_0^t \lambda(u,T_{j-1})\,du, \quad \forall\, t\in[0,T_{j-1}],$$

follows the d-dimensional standard Brownian motion under $\widehat{\mathbb{P}}_{T_j}$. Furthermore, the forward price $L(T_{j-1})$ admits the representation under $\widehat{\mathbb{P}}_{T_j}$, for $t \in [0, T_{j-1}]$

$$L(T_{j-1}) = L(t,T_{j-1})\,\exp\Big(\int_t^{T_{j-1}} \lambda_{j-1}(u)\cdot d\widehat{W}_u^{T_j} + \frac{1}{2}\int_t^{T_{j-1}} |\lambda_{j-1}(u)|^2 du\Big)$$

where we set $\lambda_{j-1}(u) = \lambda(u, T_{j-1})$. Since

$$I_1 = L(t,T_{j-1})\mathbb{E}_{\mathbb{P}_{T_j}}\Big(\mathbb{1}_D\exp\Big(\int_t^{T_{j-1}} \lambda_{j-1}(u)\cdot dW_u^{T_j} - \frac{1}{2}\int_t^{T_{j-1}} |\lambda_{j-1}(u)|^2 du\Big)\Big|\mathcal{F}_t\Big)$$

from the abstract Bayes rule we get $I_1 = L(t,T_{j-1})\,\widehat{\mathbb{P}}_{T_j}(D\,|\,\mathcal{F}_t)$. Arguing in much the same way as for I_2, we thus obtain

$$I_1 = L(t,T_{j-1})\,N\left(\frac{\ln L(t,T_{j-1}) - \ln\kappa + \frac{1}{2}v_j^2(t)}{v_j(t)}\right).$$

This completes the proof of the proposition. □

[8] See, for instance, the proof of the Black-Scholes formula in Musiela and Rutkowski (2005).

2.3.3 Hedging of Caps in the LLM

It is clear the replicating strategy for a cap is a simple sum of replicating strategies for caplets. Therefore, it is enough to focus on a particular caplet. Let us denote by $F_C(t,T_j)$ the forward price of the j^{th} caplet for the settlement date T_j. From (53), it is clear that

$$F_C(t,T_j) = \delta_j L(t,T_{j-1}) N\big(\widetilde{e}_1^j(t)\big) - \kappa\, N\big(\widetilde{e}_2^j(t)\big),$$

so that an application of Itô's formula yields[9]

$$dF_C(t,T_j) = \delta_j N\big(\widetilde{e}_1^j(t)\big)\, dL(t,T_{j-1}). \tag{55}$$

Let us consider the following self-financing trading strategy in the T_j-forward market. We start our trade at time 0 with $F_C(0,T_j)$ units of zero-coupon bonds.[10] At any time $t \le T_{j-1}$ we assume $\psi_t^j = N\big(\widetilde{e}_1^j(t)\big)$ positions in forward rate agreements (that is, single-period forward swaps) over the period $[T_{j-1},T_j]$. The associated gains/losses process V, in the T_j forward market,[11] satisfies[12]

$$dV_t = \delta_j \psi_t^j\, dL(t,T_{j-1}) = \delta_j N\big(\widetilde{e}_1^j(t)\big)\, dL(t,T_{j-1}) = dF_C(t,T)$$

with $V_0 = 0$. Consequently,

$$F_C(T_{j-1},T_j) = F_C(0,T_j) + \int_0^{T_{j-1}} \delta_j \psi_t^j\, dL(t,T_{j-1}) = F_C(0,T_j) + V_{T_{j-1}}.$$

It should be stressed that dynamic trading takes place on the interval $[0,T_{j-1}]$ only, the gains/losses (involving the initial investment) are incurred at time T_j, however. All quantities in the last formula are expressed in units of T_j-maturity zero-coupon bonds. Also, the caplet's payoff is known already at time T_{j-1}, so that it is completely specified by its forward price $F_C(T_{j-1},T_j) = \mathbf{FC}^{\,j}_{T_{j-1}}/B(T_{j-1},T_j)$. Therefore the last equality makes it clear that the strategy ψ introduced above does indeed replicate the j^{th} caplet.

It should be observed that formally the replicating strategy has also the second component, η_t^j say, which represents the number of forward contracts on T_j-maturity bond, with the settlement date T_j. Since obviously $F_B(t,T_j,T_j) = 1$ for every $t \le T_j$, so that $dF_B(t,T_j,T_j) = 0$, for the T_j-forward value of our strategy, we get $\widetilde{V}_t(\psi^j,\eta^j) = \eta_t^j = F_C(t,T_j)$ and

$$d\widetilde{V}_t(\psi^j,\eta^j) = \psi_t^j \delta_j\, dL(t,T_{j-1}) + \eta_t^j\, dF_B(t,T_j,T_j) = \delta_j N\big(\widetilde{e}_1^j(t)\big)\, dL(t,T_{j-1}).$$

It should be stressed however, with the exception for the initial investment at time 0 in T_j-maturity bonds, no bonds trading is required for the caplet's replication. In practical terms, the hedging of a cap within the framework of the LLM in done exclusively through dynamic trading in the underlying single-period swaps. Of course, the same remarks (and similar calculations) apply also to floors. In this interpretation, the component η^j simply represents the future (i.e., as of time T_{j-1}) effects of a continuous trading in forward contracts.

Alternatively, the hedging of a cap can be done in the spot (i.e., cash) market, using two simple portfolios of bonds. Indeed, it is easily seen that for the process

$$V_t(\psi^j,\eta^j) = B(t,T_{j-1})\widetilde{V}_t(\psi^j,\eta^j) = \mathbf{FC}^{\,j}_t$$

we have

$$V_t(\psi^j,\eta^j) = \psi_t^j\big(B(t,T_{j-1}) - B(t,T_j)\big) + \eta_t^j\, dF_B(t,T_j,T_j)$$

[9] The calculations here are essentially the same as in the classic Black-Scholes model.

[10] We need thus to invest $\mathbf{FC}^{\,j}_0 = F_C(0,T_j)B(0,T_j)$ of cash at time 0.

[11] That is, with the value expressed in units of T_j-maturity zero-coupon bonds.

[12] To get a more intuitive insight in this formula, it is advisable to consider first a discretized version of ψ.

and

$$\begin{aligned} dV_t(\psi^j, \eta^j) &= \psi^j_t \, d\big(B(t, T_{j-1}) - B(t, T_j)\big) + \eta^j_t \, dB(t, T_j) \\ &= N\big(\widetilde{e}^j_1(t)\big) \, d\big(B(t, T_{j-1}) - B(t, T_j)\big) + \eta^j_t \, dB(t, T_j). \end{aligned}$$

This means that the components ψ^j and η^j now represent the number of units of portfolios $B(t, T_{j-1}) - B(t, T_j)$ and $B(t, T_j)$ held at time t.

2.3.4 Bond Options in the LLM

We shall now give the bond option valuation formula within the framework of the LLM. This result was first obtained by Rady and Sandmann (1994), who adopted the PDE approach and who worked in a different setup (see also Goldys (1997), Miltersen et al. (1997), and Rady (1997)). In the present framework, it is an immediate consequence of (53) combined with (43).

Proposition 2.6 *The price* $\mathbf{C}_t$ *at time* $t \le T_{j-1}$ *of a European call option, with expiration date* T_{j-1} *and strike price* $0 < K < 1$, *written on a zero-coupon bond maturing at* $T_j = T_{j-1} + \delta_j$, *equals*

$$\mathbf{C}_t = (1 - K) B(t, T_j) N\big(l^j_1(t)\big) - K(B(t, T_{j-1}) - B(t, T_j)) N\big(l^j_2(t)\big), \tag{56}$$

where

$$l^j_{1,2}(t) = \frac{\ln\left((1-K)B(t,T_j)\right) - \ln\left(K\big(B(t,T_{j-1}) - B(t,T_j)\big)\right) \pm \frac{1}{2}\widetilde{v}_j(t)}{\widetilde{v}_j(t)}$$

and

$$\widetilde{v}^2_j(t) = \int_t^{T_{j-1}} |\lambda(u, T_{j-1})|^2 du.$$

In view of (56), it is apparent that the replication of the bond option using the underlying bonds of maturity T_{j-1} and T_j is rather involved. This should be contrasted with the case of the Gaussian Heath-Jarrow-Morton model[13] in which hedging of bond options with the use of the underlying bonds is straightforward. This illustrates the general feature that each particular way of modelling the term structure is tailored to the specific class of derivatives and hedging instruments.

2.4 Exotic Products

In all examples given below, a particular LIBOR derivative can be expressed as a single payoff of the form $Y = g(L(T_1), \ldots, L(T_n))$, which is settled at time T_n.

Knock-out cap. A path-independent example: the i^{th} caplet gets knocked out when $L(T_i)$ is below (or above) certain level (this product is a combination of a LIBOR cap and a digital). A path-dependent example: at the first fixing date T_i such that $L(T_i) \le \kappa_0$ all remaining caplets get knocked out. Thus the payment at time T_{i+1} equals

$$C_{T_{i+1}} = \delta_{i+1}(L(T_i) - \kappa)^+ \mathbb{1}_{\{\min(L(T_1), \ldots, L(T_i)) > \kappa_0\}}.$$

Asian cap. The payoff at time $T = T_n$ of an Asian cap equals

$$C_T = \left(\sum_{i=1}^n \delta_i \big(L(T_{i-1}) - \kappa\big) \right)^+.$$

Periodic cap. The floating rate coupon κ_i for payment at T_{i+1} is set at spot LIBOR, subject to it not exceeding the previously set coupon be a prescribed amount α_i. Thus κ_i equals

$$\kappa_i = \max\left(L(T_i), \kappa_{i-1} + \alpha_i\right).$$

Periodic caps are embedded in a periodically capped floating-rate note.

Flexible cap. The cap knocks out as soon as m of the caplets end up in the money. A number $m \le n$ is given in advance.

[13] In such a model the forward prices prices of bonds follow lognormal processes.

2.5 Dynamics of LIBORs and Bond Prices

We assume that the volatilities of processes $L(\cdot, T_j)$ follow deterministic functions. Put another way, we place ourselves within the framework of the lognormal LIBOR model. It is interesting to note that in all approaches, there is a uniquely determined correspondence between forward measures (and forward Brownian motions) associated with different dates $T_0, T_1, \dots, T_n$. On the other hand, however, there is a considerable degree of flexibility in the specification of the spot martingale measure. Consequently, the futures LIBOR $L^f(\cdot, T_j)$, which equals (cf. Section 2.1.3)

$$L^f(t, T_j) = \mathbb{E}_{\mathbb{P}^*}(L(T_j, T_j) \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(\widetilde{L}(T_j, T_j) \,|\, \mathcal{F}_t), \tag{57}$$

is not necessarily determined in the same way in various approaches to the lognormal model of forward LIBORs. For this reason, we start by examining the distributional properties of forward LIBORs, which are identical in all abovementioned models.

For a given function $g : \mathbb{R} \to \mathbb{R}$ and a fixed date $u \le T_j$, we are interested in the following payoff of the form $X = g\big(L(u, T_j)\big)$ which settles at time T_j. Generally speaking, to value the claim $X = g(L(u, T_j)) = \widetilde{g}(F_B(u, T_{j+1}, T_j))$ which settles at time T_j we may use the formula

$$\pi_t(X) = B(t, T_j)\mathbb{E}_{\mathbb{P}_{T_j}}(X \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T_j].$$

It is thus clear that to value a claim in the case $u \le T_j$, it is enough to know the dynamics of either $L(\cdot, T_j)$ or $F_B(\cdot, T_{j+1}, T_j)$ under the forward probability measure $\mathbb{P}_{T_j}$. If $u = T_j$, we may equally well use the the dynamics, under $\mathbb{P}_{T_j}$, of either $\widetilde{L}(\cdot, T_j)$ or $L^f(\cdot, T_j)$. For instance,

$$\pi_t(X_1) = B(t, T_j)\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j, T_{j+1}) \,|\, \mathcal{F}_t) = B(t, T_j)\mathbb{E}_{\mathbb{P}_{T_j}}(F_B^{-1}(T_j, T_{j+1}, T_j) \,|\, \mathcal{F}_t),$$

but also

$$\pi_t(X_1) = B(t, T_j)\big(1 + \delta_{j+1}\mathbb{E}_{\mathbb{P}_{T_j}}(Z(T_j) \,|\, \mathcal{F}_t)\big),$$

where we write $Z(T_j) = L(T_j, T_j) = \widetilde{L}(T_j, T_j) = L^f(T_j, T_j)$.

2.5.1 Dynamics of $L(\cdot, T_j)$ under $\mathbb{P}_{T_j}$

We shall now derive the transition probability density function (p.d.f.) of the process $L(\cdot, T_j)$ under the forward probability measure $\mathbb{P}_{T_j}$. Let us first prove the following related result, due to Jamshidian (1997).

Proposition 2.7 *Let $t \le u \le T_j$. Then*

$$\mathbb{E}_{\mathbb{P}_{T_j}}\big(L(u, T_j) \,|\, \mathcal{F}_t\big) = L(t, T_j) + \frac{\delta_{j+1}\mathrm{Var}_{\mathbb{P}_{T_{j+1}}}\big(L(u, T_j) \,|\, \mathcal{F}_t\big)}{1 + \delta_{j+1}L(t, T_j)}. \tag{58}$$

In the case of the lognormal LIBOR model, we have

$$\mathbb{E}_{\mathbb{P}_{T_j}}\big(L(u, T_j) \,|\, \mathcal{F}_t\big) = L(t, T_j)\left(1 + \frac{\delta_{j+1}L(t, T_j)\big(e^{v_j^2(t,u)} - 1\big)}{1 + \delta_{j+1}L(t, T_j)}\right), \tag{59}$$

where

$$v_j^2(t, u) = \mathrm{Var}_{\mathbb{P}_{T_{j+1}}}\Big(\int_t^u \lambda(s, T_j) \cdot dW_s^{T_{j+1}}\Big) = \int_t^u |\lambda(s, T_j)|^2\, ds. \tag{60}$$

In particular, the modified forward LIBOR rate $\widetilde{L}(t, T_j)$ satisfies[14]

$$\widetilde{L}(t, T_j) = \mathbb{E}_{\mathbb{P}_{T_j}}\big(L(T_j, T_j) \,|\, \mathcal{F}_t\big) = L(t, T_j)\left(1 + \frac{\delta_{j+1}L(t, T_j)\big(e^{v_j^2(t,T_j)} - 1\big)}{1 + \delta_{j+1}L(t, T_j)}\right).$$

[14]This equality can be referred to as the *convexity correction* for the LIBOR in arrears.

Proof. Combining (6) with the martingale property of the process $L(\cdot, T_j)$ under $\mathbb{P}_{T_{j+1}}$, we obtain

$$\mathbb{E}_{\mathbb{P}_{T_j}}\big(L(u,T_j)\,|\,\mathcal{F}_t\big) = \frac{\mathbb{E}_{\mathbb{P}_{T_{j+1}}}\big((1+\delta_{j+1}L(u,T_j))L(u,T_j)\,|\,\mathcal{F}_t\big)}{1+\delta_{j+1}L(t,T_j)}$$

so that

$$\mathbb{E}_{\mathbb{P}_{T_j}}\big(L(u,T_j)\,|\,\mathcal{F}_t\big) = L(t,T_j) + \frac{\delta_{j+1}\,\mathbb{E}_{\mathbb{P}_{T_{j+1}}}\big((L(u,T_j)-L(t,T_j))^2\,|\,\mathcal{F}_t\big)}{1+\delta_{j+1}L(t,T_j)}.$$

In the case of the lognormal LIBOR model, we have

$$L(u,T_j) = L(t,T_j)\,e^{\eta_j(t,u)-\frac{1}{2}v_j^2(t,u)},$$

where

$$\eta_j(t,u) = \int_t^u \lambda(s,T_j)\cdot dW_s^{T_{j+1}}. \tag{61}$$

Consequently,

$$\mathbb{E}_{\mathbb{P}_{T_{j+1}}}\big((L(u,T_j)-L(t,T_j))^2\,|\,\mathcal{F}_t\big) = L^2(t,T_j)\big(e^{v_j^2(t,u)}-1\big).$$

This gives the desired equality (59). The last asserted equality is a consequence of (7). □

To derive the transition probability density function (p.d.f.) of the process $L(\cdot,T_j)$, notice that for any $t \le u \le T_j$, and any bounded Borel measurable function $g:\mathbb{R}\to\mathbb{R}$ we have

$$\mathbb{E}_{\mathbb{P}_{T_j}}\big(g(L(u,T_j))\,|\,\mathcal{F}_t\big) = \frac{\mathbb{E}_{\mathbb{P}_{T_{j+1}}}\Big(g(L(u,T_j))\,\big(1+\delta_{j+1}L(u,T_j)\big)\,\Big|\,\mathcal{F}_t\Big)}{1+\delta_{j+1}L(t,T_j)}.$$

The following simple lemma appears to be useful.

Lemma 2.3 *Let ζ be a nonnegative random variable on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with the probability density function $f_{\mathbb{P}}$. Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$. Suppose that for any bounded Borel measurable function $g:\mathbb{R}\to\mathbb{R}$ we have*

$$\mathbb{E}_{\mathbb{P}}(g(\zeta)) = \mathbb{E}_{\mathbb{Q}}\big((1+\zeta)g(\zeta)\big).$$

Then the p.d.f. $f_{\mathbb{Q}}$ of ζ under $\mathbb{Q}$ satisfies $f_{\mathbb{P}}(y) = (1+y)f_{\mathbb{Q}}(y)$.

Proof. The assertion is in fact trivial since, by assumption,

$$\int_{-\infty}^{\infty} g(y)f_{\mathbb{P}}(y)\,dy = \int_{-\infty}^{\infty} g(y)(1+y)f_{\mathbb{Q}}(y)\,dy$$

for any bounded Borel measurable function $g:\mathbb{R}\to\mathbb{R}$. □

Assume the lognormal LIBOR model, and fix $x\in\mathbb{R}$. Recall that for any $t \ge u$ we have

$$L(u,T_j) = L(t,T_j)\,e^{\eta_j(t,u)-\frac{1}{2}\mathrm{Var}_{\mathbb{P}_{T_{j+1}}}(\eta_j(t,u))},$$

where $\eta_j(t,u)$ is given by (61) (so that it is independent of the σ-field $\mathcal{F}_t$). Markovian property of $L(\cdot,T_j)$ under the forward measure $\mathbb{P}_{T_{j+1}}$ is thus apparent. Denote by $p_L(t,x;u,y)$ the transition p.d.f. under $\mathbb{P}_{T_{j+1}}$ of the process $L(\cdot,T_j)$. Elementary calculations involving Gaussian densities yield

$$\begin{aligned} p_L(t,x;u,y) &= \mathbb{P}_{T_{j+1}}\{L(u,T_j)=y\,|\,L(t,T_j)=x\} \\ &= \frac{1}{\sqrt{2\pi}v_j(t,u)y}\exp\left\{-\frac{\big(\ln(y/x)+\frac{1}{2}v_j^2(t,u)\big)^2}{2v_j^2(t,u)}\right\} \end{aligned}$$

for any $x, y > 0$ and $t < u$. Taking into account Lemma 2.3, we conclude that the transition p.d.f. of the process[15] $L(\cdot, T_j)$, under the forward probability measure $\mathbb{P}_{T_j}$, satisfies

$$\widetilde{p}_L(t, x; u, y) = \mathbb{P}_{T_j}\{L(u, T_j) = y \,|\, L(t, T_j) = x\} = \frac{1 + \delta_{j+1} y}{1 + \delta_{j+1} x}\, p_L(t, x; u, y).$$

We are in a position to state the following result, which can be used, for instance, to value a contingent claim of the form $X = h(L(T_j))$ which settles at time T_j (cf. Schmidt (1996)).

Corollary 2.1 *The transition p.d.f. under $\mathbb{P}_{T_j}$ of the forward LIBOR $L(\cdot, T_j)$ equals, for any $t < u$ and $x, y > 0$,*

$$\widetilde{p}_L(t, x; u, y) = \frac{1 + \delta_{j+1} y}{\sqrt{2\pi} v_j(t, u)\, y (1 + \delta_{j+1} x)} \exp\left\{ -\frac{\big(\ln(y/x) + \frac{1}{2} v_j^2(t, u)\big)^2}{2 v_j^2(t, u)} \right\}.$$

2.5.2 Dynamics of $F_B(\cdot, T_{j+1}, T_j)$ under $\mathbb{P}_{T_j}$

Observe that the forward bond price $F_B(\cdot, T_{j+1}, T_j)$ satisfies

$$F_B(t, T_{j+1}, T_j) = \frac{B(t, T_{j+1})}{B(t, T_j)} = \frac{1}{1 + \delta_{j+1} L(t, T_j)}. \tag{62}$$

First, this implies that in the lognormal LIBOR model, the dynamics of the forward bond price $F_B(\cdot, T_{j+1}, T_j)$ are governed by the following stochastic differential equation, under $\mathbb{P}_{T_j}$,

$$dF_B(t) = -F_B(t)\big(1 - F_B(t)\big) \lambda(t, T_j) \cdot dW_t^{T_j}, \tag{63}$$

where we write $F_B(t) = F_B(t, T_{j+1}, T_j)$. If the initial condition satisfies $0 < F_B(0) < 1$, this equation can be shown to admit a unique strong solution (it satisfies $0 < F_B(t) < 1$ for every $t > 0$). This makes clear that the process $F_B(\cdot, T_{j+1}, T_j)$ – and thus also the process $L(\cdot, T_j)$ – are Markovian under $\mathbb{P}_{T_j}$. Using Corollary 2.1 and relationship (62), one can find the transition p.d.f. of the Markov process $F_B(\cdot, T_{j+1}, T_j)$ under $\mathbb{P}_{T_j}$; that is,

$$p_B(t, x; u, y) = \mathbb{P}_{T_j}\{F_B(u, T_{j+1}, T_j) = y \,|\, F_B(t, T_{j+1}, T_j) = x\}.$$

We have the following result (see Rady and Sandmann (1994), Miltersen et al. (1997), and Jamshidian (1997)).

Corollary 2.2 *The transition p.d.f. under $\mathbb{P}_{T_j}$ of the forward bond price $F_B(\cdot, T_{j+1}, T_j)$ equals, for any $t < u$ and arbitrary $0 < x, y < 1$,*

$$p_B(t, x; u, y) = \frac{x}{\sqrt{2\pi} v_j(t, u) y^2 (1 - y)} \exp\left\{ -\frac{\Big(\ln \frac{x(1-y)}{y(1-x)} + \frac{1}{2} v_j^2(t, u)\Big)^2}{2 v_j^2(t, u)} \right\}.$$

Proof. Let us fix $x \in (0, 1)$. Using (62), it is easy to show that

$$p_B(t, x; u, y) = \delta^{-1} y^{-2}\, \widetilde{p}_L\Big(t, \frac{1 - x}{\delta x}; u, \frac{1 - y}{\delta y}\Big),$$

where $\delta = \delta_{j+1}$. The formula now follows from Corollary 2.1. □

Let us observe that the results of this section can be applied to value the so-called irregular cash flows, such as caps or floors settled in advance (for more details on this issue we refer to Schmidt (1996)).

[15]The Markov property of $L(\cdot, T_j)$ under $\mathbb{P}_{T_j}$ can be easily deduced from the Markovian feature of the forward price $F_B(\cdot, T_j, T_{j+1})$ under $\mathbb{P}_{T_j}$ (see formulae (62)–(63)).

3 Modelling of Swap Rates

We shall first describe the most typical swap contracts and related options, known as *swaptions*. Subsequently, we shall present a model of forward swap rates put forward by Jamshidian (1996, 1997).

3.1 Forward Swaps

Let us consider a *forward (start) payer swap* (that is, *fixed-for-floating interest rate swap*) settled in arrears, with notional principal N_p. We consider a finite collection of dates $0 < T_0 < T_1 < \cdots < T_n$ so that $\delta_j = T_j - T_{j-1} > 0$ for every $j = 1, 2, \ldots, n$. The floating rate $L(T_{j-1})$ received at time T_j is set at time T_{j-1} by reference to the price of a zero-coupon bond over the period $[T_{j-1}, T_j]$. More specifically, $L(T_{j-1})$ is the spot LIBOR prevailing at time T_{j-1}, so that it satisfies

$$B(T_{j-1}, T_j)^{-1} = 1 + (T_j - T_{j-1})L(T_{j-1}) = 1 + \delta_j L(T_{j-1}). \tag{64}$$

Recall that in general, the forward LIBOR $L(t, T_{j-1})$ for the future time period $[T_{j-1}, T_j]$ of length δ_j satisfies

$$1 + \delta_j L(t, T_{j-1}) = \frac{B(t, T_{j-1})}{B(t, T_j)} = F_B(t, T_{j-1}, T_j), \tag{65}$$

so that $L(T_{j-1})$ coincides with $L(T_{j-1}, T_{j-1})$. At any date T_j, $j = 1, 2, \ldots, n$, the cash flows of a forward payer swap are $N_p L(T_{j-1})\delta_j$ and $-N_p \kappa \delta_j$, where κ is a preassigned fixed rate of interest (the cash flows of a forward receiver swap have the same size, but opposite signs). The number n, which coincides with the number of payments, is referred to as the *length* of a swap (for instance, the length of a 3-year swap with quarterly settlement equals $n = 12$). The dates $T_0, T_1, \ldots, T_{n-1}$ are known as *reset dates,* and the dates $T_1, T_2, \ldots, T_n$ as *settlement dates.* We shall refer to the first reset date T_0 as the *start date* of a swap. Finally, the time interval $[T_{j-1}, T_j]$ is referred to as the j^{th} *accrual period.* We may and do assume, without loss of generality, that the notional principal $N_p = 1$.

The value at time t of a forward payer swap, which is denoted by $\mathbf{FS}_t$ or $\mathbf{FS}_t(\kappa)$, equals

$$\mathbf{FS}_t(\kappa) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \sum_{j=1}^{n} \frac{B_t}{B_{T_j}} (L(T_{j-1}) - \kappa)\delta_j \,\Big|\, \mathcal{F}_t \Big\}. \tag{66}$$

Since

$$L(t, T_{j-1}) = \frac{B(t, T_{j-1}) - B(t, T_j)}{\delta_j B(t, T_j)},$$

it is clear that the process $L(\cdot, T_{j-1})$ follows a martingale under the forward martingale measure $\mathbb{P}_{T_j}$. Therefore

$$\begin{aligned}
\mathbf{FS}_t(\kappa) &= \sum_{j=1}^{n} B(t, T_j) \mathbb{E}_{\mathbb{P}_{T_j}}\big((L(T_{j-1}) - \kappa)\delta_j \,\big|\, \mathcal{F}_t \big) \\
&= \sum_{j=1}^{n} B(t, T_j)\big((L(t, T_{j-1}) - \kappa)\delta_j \big) \\
&= \sum_{j=1}^{n} \big(B(t, T_{j-1}) - B(t, T_j) - \kappa \delta_j B(t, T_j) \big).
\end{aligned}$$

After rearranging, this yields

$$\mathbf{FS}_t(\kappa) = B(t, T_0) - \sum_{j=1}^{n} c_j B(t, T_j) \tag{67}$$

for every $t \in [0, T]$, where $c_j = \kappa \delta_j$ for $j = 1, 2, \ldots, n-1$, and $c_n = \widetilde{\delta}_n = 1 + \kappa \delta_n$.

The last equality makes clear that a forward payer swap settled in arrears is, essentially, a contract to deliver a specific coupon-bearing bond and to receive in the same time a zero-coupon bond. Relationship (67) may also be established through a straightforward comparison of the future cash flows from these bonds. Note that (67) provides a simple method for the replication of a swap contract, independent of the term structure model.

In the forward payer swap settled in advance – that is, in which each reset date is also a settlement date – the discounting method varies from country to country. In some markets, the cash flows of a swap settled in advance at reset dates $T_j,\ j = 0, 1, \ldots, n-1$, are $L(T_j)\delta_{j+1}(1 + L(T_j)\delta_{j+1})^{-1}$ and $-\kappa\delta_{j+1}(1 + L(T_j)\delta_{j+1})^{-1}$. The value $\mathbf{FS}_t^{**}(\kappa)$ at time t of such a swap is

$$\begin{aligned}\mathbf{FS}_t^{**}(\kappa) &= \mathbb{E}_{\mathbb{P}^*}\Big\{\sum_{j=0}^{n-1}\frac{B_t}{B_{T_j}}\,\frac{\delta_{j+1}(L(T_j)-\kappa)}{1+\delta_{j+1}L(T_j)}\,\Big|\,\mathcal{F}_t\Big\} \\ &= \mathbb{E}_{\mathbb{P}^*}\Big\{\sum_{j=0}^{n-1}\frac{B_t}{B_{T_j}}(L(T_j)-\kappa)\delta_{j+1}B(T_j,T_{j+1})\,\Big|\,\mathcal{F}_t\Big\} \\ &= \mathbb{E}_{\mathbb{P}^*}\Big\{\sum_{j=0}^{n-1}\frac{B_t}{B_{T_{j+1}}}(L(T_j)-\kappa)\delta_{j+1}\,\Big|\,\mathcal{F}_t\Big\},\end{aligned}$$

which coincides with the value of the swap settled in arrears. This is by no means surprising, since the payoffs $L(T_j)\delta_{j+1}(1 + L(T_j)\delta_{j+1})^{-1}$ and $-\kappa\delta_{j+1}(1 + L(T_j)\delta_{j+1})^{-1}$ at time T_j are easily seen to be equivalent to payoffs $L(T_j)\delta_{j+1}$ and $-\kappa\delta_{j+1}$ respectively at time T_{j+1} (recall that $1+L(T_j)\delta_{j+1} = B^{-1}(T_j, T_{j+1})$).

3.2 Forward Swap Rates

In what follows, we shall restrict our attention to interest rate swaps settled in arrears. As mentioned, a swap agreement is worthless at initiation. This important feature of a swap leads to the following definition, which refers in fact to the more general concept of a forward swap. Basically, a forward swap rate is that fixed rate of interest which makes a forward swap worthless.

Definition 3.1 The *forward swap rate* $\kappa(t, T_0, n)$ at time t for the date T_0 is that value of the fixed rate κ which makes the value of the forward swap zero, i.e., that value of κ for which $\mathbf{FS}_t(\kappa) = 0$. Using (67), we obtain

$$\kappa(t, T_0, n) = (B(t, T_0) - B(t, T_n))\Big(\sum_{j=1}^{n}\delta_j B(t, T_j)\Big)^{-1}. \tag{68}$$

Note that the definition of a forward swap rate implicitly refers to a swap contract of length n which starts at time T_0. It would thus be more correct to refer to $\kappa(t, T_0, n)$ as the *n-period forward swap rate* prevailing at time t, for the future date T_0. A forward swap rate is a rather theoretical concept, as opposed to swap rates, which are quoted daily (subject to an appropriate bid-ask spread) by financial institutions who offer interest rate swap contracts to their institutional clients. In practice, swap agreements of various lengths are offered. Also, typically, the length of the reference period varies over time; for instance, a 5-year swap may be settled quarterly during the first three years, and semi-annually during the last two.

Finally, it will be useful to express that value at time t of a given forward swap with fixed rate κ in terms of the current value of the forward swap rate. Since obviously $\mathbf{FS}_t(\kappa(t, T_0, n)) = 0$, using (67), we get

$$\mathbf{FS}_t(\kappa) = \mathbf{FS}_t(\kappa) - \mathbf{FS}_t(\kappa(t, T_0, n)) = \sum_{j=1}^{n}(\kappa(t, T_0, n) - \kappa)\delta_j B(t, T_j). \tag{69}$$

3.3 Models of Co-Terminal Swap Rates

Modelling of co-terminal forward swap rates was examined by Jamshidian (1996, 1997). We assume, as before, that the tenor structure $0 < T_0 < T_1 < \cdots < T_n$ is given. Recall that $\delta_j = T_j - T_{j-1}$ for $j = 1, 2, \ldots, n$, and thus $T_j = \sum_{i=0}^{j} \delta_i$ for every $j = 0, 1, \ldots, n$.

For any fixed j, we consider a fixed-for-floating forward (payer) swap which starts at time T_j and has $n - j$ accrual periods, whose consecutive lengths are $\delta_{j+1}, \ldots, \delta_n$. The last settlement date is thus T_n for every swap considered here. This specific feature justifies the name of *co-terminal* or *fixed-maturity* forward swap rates.

The fixed interest rate paid at each of reset dates T_l for $l = j + 1, j + 2, \ldots, n$ equals κ, and the corresponding floating rate, $L(T_l)$, is found using the formula

$$B(T_l, T_{l+1})^{-1} = 1 + (T_{l+1} - T_l)L(T_l) = 1 + \delta_{l+1}L(T_l),$$

i.e., it coincides with the LIBOR rate $L(T_l, T_l)$. It is not difficult to check, using no-arbitrage arguments, that the value of such a swap equals, for $t \in [0, T_j]$ (by convention, the notional principal equals 1)

$$\mathbf{FS}_t(\kappa) = B(t, T_j) - \sum_{l=j+1}^{n} c_l B(t, T_l),$$

where $c_l = \kappa\delta_l$ for $l = j + 1, j + 2, \ldots, n - 1$, and $c_n = 1 + \kappa\delta_n$. The associated *forward swap rate,* $\kappa(t, T_j, n - j)$, that is, that value of a fixed rate κ for which such a swap is worthless at time t, is given by the formula

$$\kappa(t, T_j, n - j) = \frac{B(t, T_j) - B(t, T_n)}{\delta_{j+1}B(t, T_{j+1}) + \cdots + \delta_n B(t, T_n)} = \frac{B(t, T_j) - B(t, T_n)}{G_t(n - j)} \tag{70}$$

for every $t \in [0, T_j]$, $j = 0, 1, \ldots, n - 1$. In this section, we consider the family of forward swap rates $\widetilde{\kappa}(t, T_j) = \kappa(t, T_j, n - j)$ for $j = 0, 1, \ldots, n - 1$. Let us stress that the underlying swap agreements differ in length, but they all have a common expiration date, T_n. The process $G(n - j)$ is termed the *level process* or the *present value of the basis point* (PVBP, for short).

3.3.1 Forward Swap Measures

The *forward swap measure* $\widetilde{\mathbb{P}}_{T_{j+1}}$ (sometimes termed the *level measure*) is a probability measure equivalent to $\mathbb{P}$ such that the relative bond prices $B(t, T_k)/G_t(n-j)$, $k = 0, 1, \ldots, n$, are martingales under $\widetilde{\mathbb{P}}_{T_{j+1}}$. We denote $\widetilde{\kappa}(t, T_j) = \kappa(t, T_j, n - j)$. Let us set, for every $1 \leq i \leq j \leq n$,

$$g_t^{ij} = \sum_{k=j}^{n-1} \delta_{k+1} \prod_{l=i+1}^{k} (1 + \delta_l \widetilde{\kappa}(t, T_l)).$$

One can show by induction that $G_t(n - j) = B(t, T_n)g_t^{jj}$. Observe that $\widetilde{\mathbb{P}}_{T_n} = \mathbb{P}_{T_n}$ is the forward measure for the date T_n. We may thus set $\widetilde{W}^{T_n} = W^{T_n} = W$ for some d-dimensional Brownian motion W.

Proposition 3.1 *Assume that for every* $j = 0, 1, \ldots, n - 1$ *we have*

$$d\widetilde{\kappa}(t, T_j) = \mu_t^j \, dt + \phi_t^j \cdot dW_t.$$

Then the forward swap rates $\widetilde{\kappa}(t, T_j)$, $j = 0, 1, \ldots, n - 1$ *satisfy the following recursive relationship*

$$d\widetilde{\kappa}(t, T_j) = -\sum_{k=j+1}^{n-1} \frac{\delta_k g_t^{jk} \phi_t^j \phi_t^k}{(1 + \delta_k \widetilde{\kappa}(t, T_k))g_t^{jj}} \, dt + \phi_t^j \cdot dW_t. \tag{71}$$

3.3.2 Lognormal Model of Co-Terminal Swap Rates

In the special case of the lognormal model of co-terminal forward swap rates, Jamshidian's construction can be summarized as follows. Assume that

$$d\widetilde{\kappa}(t,T_j) = \mu_t^j\,dt + \nu(t,T_j)\widetilde{\kappa}(t,T_j)\cdot dW_t$$

so that $\phi_t^j = \nu(t,T_j)\widetilde{\kappa}(t,T_j)$. Using the recursive relationship (71), we obtain

$$\widetilde{\kappa}(t,T_n) = \widetilde{\kappa}(0,T_n)\,\mathcal{E}_t\left(\int_0^{\cdot} \nu(u,T_n)\cdot dW_u\right),$$

and for every $j = 0,1,\dots n-1$

$$\widetilde{\kappa}(t,T_j) = \widetilde{\kappa}(0,T_j)h_t^j\,\mathcal{E}_t\left(\int_0^{\cdot} \nu(u,T_j)\cdot dW_u\right),$$

where the process h_t^j is given by the following expression

$$h_t^j = \exp\left(-\int_0^t \sum_{k=j+1}^{n-1} \frac{\delta_k g_u^{jk}\nu(u,T_j)\nu(u,T_k)\widetilde{\kappa}(u,T_k)}{(1+\delta_k\widetilde{\kappa}(u,T_k))g_u^{jj}}\,du\right).$$

Proposition 3.2 *For every $j = 0,1,\dots,n-1$, the forward swap rate $\widetilde{\kappa}(\cdot,T_j)$ satisfies the following SDE*

$$d\widetilde{\kappa}(t,T_j) = \widetilde{\kappa}(t,T_j)\nu(t,T_j)\cdot d\widetilde{W}_t^{T_{j+1}}, \tag{72}$$

where $\widetilde{W}^{T_{j+1}}$ follows a standard d-dimensional Brownian motion under the corresponding forward swap measure $\widetilde{\mathbb{P}}_{T_{j+1}}$.

Remarks. It should be noted that lognormal model of co-terminal forward swap rates and the lognormal model of forward LIBORs are incompatible with each other. Indeed, it is not difficult to check that the forward LIBORs and swap rates satisfy

$$\kappa(t,T_j,n-j) = \frac{\prod_{i=j}^{n-1}(1+\delta_{i+1}L(t,T_i)) - 1}{\sum_{i=j}^{n}\delta_i\prod_{k=i+1}^{n-1}(1+\delta_{k+1}L(t,T_k))}.$$

The formula above shows that LIBORs and swap rates cannot have simultaneously deterministic volatilities. We conclude that the market (i.e., lognormal) models for LIBORs and swap rates are inconsistent with each other.

3.4 Valuation of Swaptions

For a long time, Black's swaptions formula was merely a (widely used) practical tool to value swaptions. Indeed, the use of this formula was not supported by the existence of a reliable term structure model. The formal derivation of this heuristic results within the framework of a well established term structure model was first achieved in Jamshidian (1997).

3.4.1 Payer and Receiver Swaptions

The owner of a *payer* (*receiver,* respectively) *swaption* with strike rate κ, maturing at time $T = T_0$, has the right to enter at time T the underlying forward payer (receiver, respectively) swap settled in arrears.[16] Because $\mathbf{FS}_T(\kappa)$ is the value at time T of the payer swap with the fixed interest rate κ, it is clear that the price of the payer swaption at time t equals

$$\mathbf{PS}_t = \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\Big(\mathbf{FS}_T(\kappa)\Big)^+\,\Big|\,\mathcal{F}_t\Big\}.$$

[16]By convention, the notional principal of the underlying swap (and thus also the notional principal of the swaption) equals $N_p = 1$.

Using (66), we obtain

$$\mathbf{PS}_t = \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\Big(\mathbb{E}_{\mathbb{P}^*}\Big(\sum_{j=1}^{n}\frac{B_T}{B_{T_j}}(L(T_{j-1})-\kappa)\delta_j\,\Big|\,\mathcal{F}_T\Big)\Big)^+\,\Big|\,\mathcal{F}_t\Big\}. \tag{73}$$

On the other hand, in view of (69) we also have

$$\mathbf{PS}_t = \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\Big(\mathbb{E}_{\mathbb{P}^*}\Big(\sum_{j=1}^{n}\frac{B_T}{B_{T_j}}(\kappa(T,T,n)-\kappa)\delta_j\,\Big|\,\mathcal{F}_T\Big)\Big)^+\,\Big|\,\mathcal{F}_t\Big\} \tag{74}$$

The last equality yields

$$\begin{aligned}
\mathbf{PS}_t &= \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\Big(\mathbb{E}_{\mathbb{P}^*}\Big(\sum_{j=1}^{n}\frac{B_T}{B_{T_j}}(\kappa(T,T,n)-\kappa)\delta_j\,\Big|\,\mathcal{F}_T\Big)\Big)^+\,\Big|\,\mathcal{F}_t\Big\}\\
&= \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\mathbb{E}_{\mathbb{P}^*}\Big(\sum_{j=1}^{n}\frac{B_T}{B_{T_j}}(\kappa(T,T,n)-\kappa)^+\delta_j\,\Big|\,\mathcal{F}_T\Big)\,\Big|\,\mathcal{F}_t\Big\}\\
&= \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\sum_{j=1}^{n}\delta_j B(T,T_j)\mathbb{E}_{\mathbb{P}_{T_j}}\big((\kappa(T,T,n)-\kappa)^+\,\big|\,\mathcal{F}_T\big)\,\Big|\,\mathcal{F}_t\Big\}\\
&= \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\sum_{j=1}^{n}\delta_j B(T,T_j)(\kappa(T,T,n)-\kappa)^+\,\Big|\,\mathcal{F}_t\Big\}\\
&= \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\Big(1-\sum_{j=1}^{n}c_j B(T,T_j)\Big)^+\,\Big|\,\mathcal{F}_t\Big\}.
\end{aligned}$$

Similarly, for the receiver swaption, we have

$$\mathbf{RS}_t = \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\Big(-\mathbf{FS}_T(\kappa)\Big)^+\,\Big|\,\mathcal{F}_t\Big\},$$

that is

$$\mathbf{RS}_t = \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\Big(\mathbb{E}_{\mathbb{P}^*}\Big(\sum_{j=1}^{n}\frac{B_T}{B_{T_j}}(\kappa-L(T_{j-1}))\delta_j\,\Big|\,\mathcal{F}_T\Big)\Big)^+\,\Big|\,\mathcal{F}_t\Big\}, \tag{75}$$

where we write $\mathbf{RS}_t$ to denote the price at time t of a receiver swaption. Consequently, reasoning in much the same way as in the case of a payer swaption, we get

$$\begin{aligned}
\mathbf{RS}_t &= \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\Big(\mathbb{E}_{\mathbb{P}^*}\Big(\sum_{j=1}^{n}\frac{B_T}{B_{T_j}}(\kappa-\kappa(T,T,n))\delta_j\,\Big|\,\mathcal{F}_T\Big)\Big)^+\,\Big|\,\mathcal{F}_t\Big\}\\
&= \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\mathbb{E}_{\mathbb{P}^*}\Big(\sum_{j=1}^{n}\frac{B_T}{B_{T_j}}(\kappa-\kappa(T,T,n))^+\delta_j\,\Big|\,\mathcal{F}_T\Big)\,\Big|\,\mathcal{F}_t\Big\}\\
&= \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t}{B_T}\Big(\sum_{j=1}^{n}c_j B(T,T_j)-1\Big)^+\,\Big|\,\mathcal{F}_t\Big\}.
\end{aligned}$$

We shall first focus on a payer swaption. In view of (73), it is apparent that a payer swaption is exercised at time T if and only if the value of the underlying swap is positive at this date. It should be made clear that a swaption may be exercised by its owner only at its maturity date T. If exercised, a swaption gives rise to a sequence of cash flows at prescribed future dates. By considering the future cash flows from a swaption and from the corresponding *market swap*[17] available at time

[17] At any time t, a *market swap* is that swap whose current value equals zero. Put more explicitly, it is the swap in which the fixed rate κ equals the current swap rate.

T, it is easily seen that the owner of a swaption is protected against the adverse movements of the swap rate that may occur before time T. Suppose, for instance, that the swap rate at time T is greater than κ. Then by combining the swaption with a market swap, the owner of a swaption with exercise rate κ is entitled to enter at time T, at no additional cost, a swap contract in which the fixed rate is κ. If, on the contrary, the swap rate at time T is less than κ, the swaption is worthless, but its owner is, of course, able to enter a market swap contract based on the current swap rate $\kappa(T,T,n) \leq \kappa$. Concluding, the fixed rate paid by the owner of a swaption who intends to initiate a swap contract at time T will never be above the preassigned level κ.

Put-call parity for swaptions. Let mention the put-call parity relationship for swaptions. It follows easily from (73)–(75) that $\mathbf{PS}_t - \mathbf{RS}_t = \mathbf{FS}_t$, i.e.,

$$\textit{Payer Swaption}\,(t) \;-\; \textit{Receiver Swaption}\,(t) \;=\; \textit{Forward Swap}\,(t)$$

provided that both swaptions expire at the same date T (and have the same contractual features).

Swaption as an option on a swap rate. Notice that we that we have shown, in particular, that

$$\mathbf{PS}_t = \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_T} \mathbb{E}_{\mathbb{P}^*}\Big(\sum_{j=1}^{n} \frac{B_T}{B_{T_j}} (\kappa(T,T,n) - \kappa)^+ \delta_j \,\Big|\, \mathcal{F}_T \Big) \,\Big|\, \mathcal{F}_t \Big\} \tag{76}$$

This shows that a payer swaption is essentially equivalent a sequence of fixed payments $d_j^p = \delta_j(\kappa(T,T,n) - \kappa)^+$ which are received at settlement dates $T_1, T_2, \ldots, T_n$, but whose value is known already at the expiry date T. In words, a payer swaption can be seen as a specific call option on a forward swap rate, with fixed strike level κ. The exercise date of the option is T, but the payoff takes place at each date $T_1, \ldots, T_n$. This equivalence may also be derived by directly verifying that the future cash flows from the following portfolios established at time T are identical: portfolio A – a swaption and a market swap; and portfolio B – a just described call option on a swap rate and a market swap. Indeed, both portfolios correspond to a payer swap with the fixed rate equal to κ.

Swaption as an option on a coupon bond. Equality

$$\mathbf{PS}_t = \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_T} \Big(1 - \sum_{j=1}^{n} c_j B(T,T_j) \Big)^+ \,\Big|\, \mathcal{F}_t \Big\} \tag{77}$$

shows that the payer swaption may also be seen as a standard put option on a coupon-bearing bond with the coupon rate κ, with exercise date T and strike price 1.

Similar remarks are valid for the receiver swaption. In particular, a receiver swaption can also be viewed as a sequence of put options on a swap rate which are not allowed to be exercised separately. At time T the long party receives the value of a sequence of cash flows, discounted from time T_j, $j = 1, 2, \ldots, n$, to the date T, defined by $\delta_j\,(\kappa - \kappa(T,T,n))^+$. On the other hand, a receiver swaption may be seen as a call option, with strike price 1 and expiry date T, written on a coupon bond with coupon rate equal to the strike rate κ of the underlying forward swap.

3.4.2 Market Formula for Swaptions

The commonly used market formula for swaptions, based on the assumption that the underlying swap rate follows a geometric Brownian motion under the intuitively perceived "market probability" $\mathbb{Q}$, is given by *Black's swaption formula* (cf. Neuberger (1990))

$$\mathbf{PS}_t = \sum_{j=1}^{n} B(t,T_j)\delta_j \Big(\kappa(t,T,n) N\big(h_1(t,T)\big) - \kappa N\big(h_2(t,T)\big) \Big), \tag{78}$$

where $T = T_0$ is the swaption's expiry date, and

$$h_{1,2}(t,T) = \frac{\ln(\kappa(t,T,n)/\kappa) \pm \frac{1}{2}\,\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

for some constant *implied swaption volatility* $\sigma > 0$. To examine formula (78), let us assume, for simplicity, that $t = 0$. In this case, using general valuation results, we obtain the following equality

$$\mathbf{PS}_0 = \sum_{j=1}^{n} \delta_j B(0, T_j)\, \mathbb{E}_{\mathbb{P}_{T_j}} \big((\kappa(T, T, n) - \kappa)^+\big).$$

This suggests that market convention implicitly assumes lognormal probability law for the swap rate $\kappa(T, T, n)$ under $\mathbb{P}_{T_j}$. The swaption valuation formula obtained in the framework of the lognormal model of LIBORs appears to be more involved. It reduces to the "market formula" (78) only in very special circumstances.

On the other hand, the swaption price derived within the lognormal model of forward swap rates agrees with the (78). More precisely, this holds for a specific family of swaptions. This is by no means surprising, as the model was exactly tailored to handle a particular family of swaptions, or rather, to analyze certain path-dependent swaptions (such as *Bermudan swaptions*). The price of a cap in the lognormal model of swap rates is not given by a closed-form expression, however.

3.4.3 Valuation of Co-Terminal Swaptions

For a fixed, but otherwise arbitrary, date $T_j,\ j = 0, 1, \dots, n-1,$ we consider a swaption with expiry date $T_j,$ written on a forward payer swap settled in arrears. The underlying forward payer swap starts at date $T_j,$ has the fixed rate κ and $n-j$ accrual periods. Such a swaption is referred to as the j^{th} swaption in what follows. Notice that the j^{th} swaption can be seen as a contract which pays to its owner the amount $\delta_k(\kappa(T_j, T_j, n-j) - \kappa)^+$ at each settlement date T_k, where $k = j+1, j+2, \dots, n$ (recall that we assume that the notional principal $N_p = 1$). Equivalently, the j^{th} swaption pays an amount

$$\widetilde{Y} = \sum_{k=j+1}^{n} \delta_k B(T_j, T_k) \big(\widetilde{\kappa}(T_j, T_j) - \kappa\big)^+$$

at maturity date T_j. It is useful to observe that $\widetilde{Y}$ admits the following representation in terms of the numeraire process $G(n-j) = \sum_{k=j+1}^{n} \delta_k B(T_j, T_k)$

$$\widetilde{Y} = G_{T_j}(n-j) \big(\widetilde{\kappa}(T_j, T_j) - \kappa\big)^+.$$

Jamshidian's model of co-terminal forward swap rates specifies the dynamics of the process $\widetilde{\kappa}(\cdot, T_j)$ through the following SDEs (cf. (72))

$$d\widetilde{\kappa}(t, T_j) = \widetilde{\kappa}(t, T_j)\nu(t, T_j) \cdot d\widetilde{W}_t^{T_{j+1}},$$

where $\widetilde{W}^{T_{j+1}}$ follows a standard d-dimensional Brownian motion under the corresponding forward swap measure $\widetilde{\mathbb{P}}_{T_{j+1}}$. Recall that the definition of $\widetilde{\mathbb{P}}_{T_{j+1}}$ implies that any process of the form $B(t, T_k)/G_t(n-j),\ k = 0, 1, \dots, n,$ is a local martingale under $\widetilde{\mathbb{P}}_{T_{j+1}}$.

From the general considerations concerning the choice of a numeraire (see, e.g. Geman et al. (1995) or Musiela and Rutkowski (2005)) it is easy to see that the arbitrage price $\pi_t(X)$ of an attainable contingent claim $X = g(B(T_j, T_{j+1}), \dots, B(T_j, T_n))$ equals, for $t \in [0, T_j]$,

$$\pi_t(X) = G_t(n-j)\, \mathbb{E}_{\widetilde{\mathbb{P}}_{T_{j+1}}} \big(G_{T_j}^{-1}(n-j) X \,\big|\, \mathcal{F}_t\big),$$

provided that X settles at time T_j. Applying the last formula to the swaption's payoff $\widetilde{Y},$ we obtain the following representation for the arbitrage price $\mathbf{PS}_t^j$ at time $t \in [0, T_j]$ of the j^{th} swaption

$$\mathbf{PS}_t^j = \pi_t(\widetilde{Y}) = G_t(n-j)\, \mathbb{E}_{\widetilde{\mathbb{P}}_{T_{j+1}}} \big((\widetilde{\kappa}(T_j, T_j) - \kappa)^+ \,\big|\, \mathcal{F}_t\big).$$

We assume from now on that $\nu(\cdot, T_j) : [0, T_j] \to \mathbb{R}^d$ is a bounded deterministic function. In other words, we place ourselves within the framework of the lognormal model of co-terminal forward swap rates. The proof of following result, due to Jamshidian (1996, 1997), is thus straightforward.

Proposition 3.3 *For any $j = 0, 1, \ldots, n-1$, the arbitrage price at time $t \in [0, T_j]$ of the j^{th} swaption equals*

$$\mathbf{PS}^j_t = \sum_{k=j+1}^{n} \delta_k B(t, T_k) \Big(\widetilde{\kappa}(t, T_j) N\big(\widetilde{h}_1(t, T_j)\big) - \kappa N\big(\widetilde{h}_2(t, T_j)\big) \Big),$$

where N denotes the standard Gaussian cumulative distribution function, and

$$\widetilde{h}_{1,2}(t, T_j) = \frac{\ln(\widetilde{\kappa}(t, T_j)/\kappa) \pm \frac{1}{2} v^2(t, T_j)}{v(t, T_j)},$$

with $v^2(t, T_j) = \int_t^{T_j} |\nu(u, T_j)|^2 \, du$.

3.4.4 Hedging of Co-Terminal Swaptions

The replicating strategy for a swaption within the present framework has similar features as the replicating strategy for a cap in the lognormal model of forward LIBOR rates. Therefore, we shall focus mainly on differences between these two cases. Let us fix j, and let us denote by $F_{S^j}(t, T)$ the relative price at time $t \le T_j$ of the j^{th} swaption, when the level process

$$G_t(n - j) = \sum_{k=j+1}^{n} \delta_k B(t, T_k)$$

is chosen as a numeraire asset. From Proposition 3.3, we find easily that for every $t \le T_j$

$$F_{S^j}(t, T_j) = \widetilde{\kappa}(t, T_j) N\big(\widetilde{h}_1(t, T_j)\big) - \kappa N\big(\widetilde{h}_2(t, T_j)\big).$$

Applying Itô's formula to the last expression, we obtain

$$dF_{S^j}(t, T_j) = N\big(\widetilde{h}_1(t, T_j)\big) \, d\widetilde{\kappa}(t, T_j). \tag{79}$$

Let us consider the following self-financing trading strategy. We start our trade at time 0 with the amount $\mathbf{PS}^j_0$ of cash, which is then immediately invested in the portfolio $G(n-j)$.[18] At any time $t \le T_j$ we assume $\psi^j_t = N\big(h_1(t, T_j)\big)$ positions in market forward swaps (of course, these swaps have the same starting date and tenor structure as the underlying forward swap). The associated gains/losses process V, expressed in units of the numeraire asset $G(n-j)$, satisfies

$$dV_t = \psi^j_t \, d\widetilde{\kappa}(t, T_j) = N\big(\widetilde{h}_1(t, T_j)\big) \, d\widetilde{\kappa}(t, T_j) = dF_{S^j}(t, T_j)$$

with $V_0 = 0$. Consequently,

$$F_{S^j}(T_j, T_j) = F_{S^j}(0, T_j) + \int_0^{T_j} \psi^j_t \, d\widetilde{\kappa}(t, T_j) = F_{S^j}(0, T_j) + V_{T_j}.$$

Here the dynamic trading in market forward swaps takes place at any date $t \in [0, T_j]$, and all gains/losses from trading (involving the initial investment) are expressed in units of $G(n-j)$. The last equality makes it clear that the strategy ψ^j introduced above does indeed replicate the j^{th} swaption.

[18] One unit of portfolio $G(n-j)$ costs $\sum_{k=j+1}^{n} \delta_k B(0, T_k)$ at time 0.

3.5 Bermudan Swaptions

A Bermudan receiver swaption is the option, which at each given date $t_j \le T_j$ gives the holder the right to enter the j^{th} swap, provided this right has not yet been exercised at a previous date $t_l,\ l = 0, 1, \dots, j-1$. Bermudan swaptions frequently arise as embedded options in cancellable (callable) swaps. As expected, the valuation and hedging of a Bermudan swaption is closely related to an optimal stopping problem. Notice that the j^{th} swap is worth at time t_j the amount

$$G_{t_j}(n-j)(\kappa - \widetilde{\kappa}(t_j, T_j)).$$

An equivalent payoff at time T_n is

$$V^j = G_{t_j}(n-j)(\kappa - \widetilde{\kappa}(t_j, T_j))/B(t_j, T_n).$$

If a holder exercises the Bermudan swaption at time t_j, he will receive V^j at time T_n. Let us define inductively a sequence C^j, $j = 1, \dots, n$ of random variables by setting $C^n = \max(V^n, 0)$ and

$$C^j = 1\!\!1_{\{V^j \ge \mathbb{E}_{\widetilde{\mathbb{P}}_{T_n}}(C^{j+1}\,|\,\mathcal{F}_{t_j})\}} V^j + 1\!\!1_{\{V^j < \mathbb{E}_{\widetilde{\mathbb{P}}_{T_n}}(C^{j+1}\,|\,\mathcal{F}_{t_j})\}} C^{j+1}.$$

In view of the optimality assumption, the payoff of the Bermudan swaption at time T_n equals C^1. Consequently, the price at time $t \le t_0$ of the contract can be found by evaluating the conditional expectation $B(t, T_n)\, \mathbb{E}_{\widetilde{\mathbb{P}}_{T_n}}(C^1\,|\,\mathcal{F}_t)$.

3.6 Choice of Numeraire Portfolio

Let us consider two particular portfolios of zero-coupon bonds, with value processes V_t^1 and V_t^2. Typically, we are interested in options to exchange one of these portfolios for another, at a given date T. Let us write

$$C_T = (V_T^1 - KV_T^2)^+ = V_T^1\, 1\!\!1_D - KV_T^2\, 1\!\!1_D, \tag{80}$$

where $K > 0$ is a constant, and $D = \{V_T^1 > KV_T^2\}$ is the exercise set. It is easy to check using the abstract Bayes rule that the equality

$$\frac{d\mathbb{P}^1}{d\mathbb{P}^2} = \frac{V_0^2}{V_0^1}\,\frac{V_T^1}{V_T^2}, \quad \mathbb{P}^2\text{-a.s.}, \tag{81}$$

links the martingale measures $\mathbb{P}^1$ and $\mathbb{P}^2$ associated with the choice of value processes V^1 and V^2 as discount factors, respectively (both probability measures are considered here on $(\Omega, \mathcal{F}_T)$). Furthermore, the arbitrage price of the option admits the following representation

$$C_t = V_t^1 \mathbb{P}^1(D\,|\,\mathcal{F}_t) - KV_t^2 \mathbb{P}^2(D\,|\,\mathcal{F}_t), \quad \forall\, t \in [0, T]. \tag{82}$$

To obtain the Black-Scholes like formula for the option's price C_t, it is enough to assume that the relative price V^1/V^2 follows a lognormal martingale under $\mathbb{P}^2$, so that

$$d\,(V_t^1/V_t^2) = (V_t^1/V_t^2)\gamma_t^{1,2} \cdot dW_t^{1,2} \tag{83}$$

for some deterministic function $\gamma^{1,2} : [0, T] \to \mathbb{R}^d$ (for simplicity, one may also assume that the function $\gamma^{1,2}$ is bounded), where $W^{1,2}$ follows a standard Brownian motion under $\mathbb{P}^2$. In view of (81), the Radon-Nikodým density of $\mathbb{P}^1$ with respect to $\mathbb{P}^2$ equals

$$\frac{d\mathbb{P}^1}{d\mathbb{P}^2} = \mathcal{E}_T\Big(\int_0^{\cdot} \gamma_u^{1,2} \cdot dW_u^{1,2}\Big), \quad \mathbb{P}^2\text{-a.s.}, \tag{84}$$

and thus the process

$$W_t^{2,1} = W_t^{1,2} - \int_0^t \gamma_u^{1,2}\, du, \quad \forall\, t \in [0, T],$$

is a standard Brownian motion under $\mathbb{P}^1$. Reasoning in the same way as in the proof of the classic Black-Scholes formula, we obtain

$$C_t = V_t^1 N\big(d_1(t,T)\big) - K V_t^2 N\big(d_2(t,T)\big), \tag{85}$$

where

$$d_{1,2}(t,T) = \frac{\ln(V_t^1/V_t^2) - \ln K \pm \frac{1}{2}\, v_{1,2}^2(t,T)}{v_{1,2}(t,T)}$$

and

$$v_{1,2}^2(t,T) = \int_t^T |\gamma_u^{1,2}|^2\, du, \quad \forall\, t \in [0,T].$$

Of course, the caps and swaptions valuation formulae in lognormal models described above can be seen as special cases of (85). For the j^{th} caplet, we take

$$V_t^1 = B(t,T_j) - B(t,T_{j+1}), \quad V_t^2 = \delta_{j+1} B(t,T_{j+1}).$$

In the case of the j^{th} swaption, we have

$$V_t^1 = B(t,T_j) - B(t,T_n), \quad V_t^2 = \sum_{k=j+1}^{n} \delta_k B(t,T_k).$$

Of course, the idea of a change of a numeraire can be applied to numerous other interest rate derivatives.

It is worthwhile to notice that in order to get the valuation result (85) for $t=0$, it is enough to assume that the random variable V_T^1/V_T^2 has a lognormal probability law under the martingale measure $\mathbb{P}^2$. This simple observation underpins the construction of the so-called *Markov-functional interest rate models* – this alternative approach to term structure modelling was developed by Hunt et al. (1996, 2000).

A more straightforward generalization of lognormal models of the term structure was developed by Andersen and Andreasen (2000). In this case, the assumption that the volatility is deterministic is replaced by a suitable functional form of the volatility. The resulting models are capable to handle the so-called *volatility skew* in observed option prices (empirical studies have shown that the implied volatilities of observed caps and swaptions prices tend to be decreasing functions of the strike level). The main focus in Andersen and Andreasen (1997) is on the use of the CEV process[19] as a model of the forward LIBOR rate. Put more explicitly, they generalize equality (19) by postulating that

$$dL(t,T_j) = L^{\alpha}(t,T_j)\, \lambda(t,T_j) \cdot dW_t^{T_{j+1}}, \quad \forall\, t \in [0,T_j],$$

where $\alpha > 0$ is a strictly positive constant. They derive closed-form solutions for caplet prices under the above specification of the dynamics of LIBOR rates with $\alpha \neq 1$, in terms of the cumulative distribution function of a non-central χ^2 probability law. It appears that depending on the choice of the parameter α, the implied Black's volatilities of caplet prices, considered as a function of the strike level $\kappa > 0$, exhibit downward- or upward-sloping skew.

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